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- Application: "harvesting" a given DFT pair $\mathbf{x} \longleftrightarrow \mathbf{X}$

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- Circular shift $\mathbf{P}$
- Circular reversal $\mathbf{R}$
- The diagonal matrix $\mathbf{F}$
- Application: "harvesting" a given DFT pair $\mathbf{x} \longleftrightarrow \mathbf{X}$, i.e., deriving further pairs without full DFT computation

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- Shift is downward:

  $$[\,0 \;\; 1 \;\; \ldots \;\; N-2 \;\; N-1\,]^T$$

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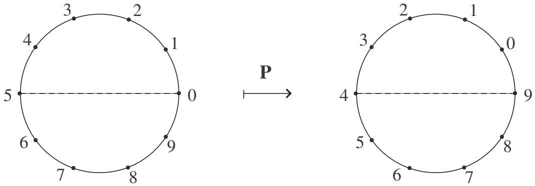

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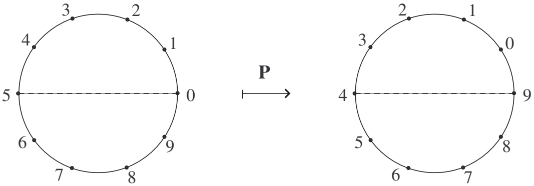

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$$\mathbf{P} = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

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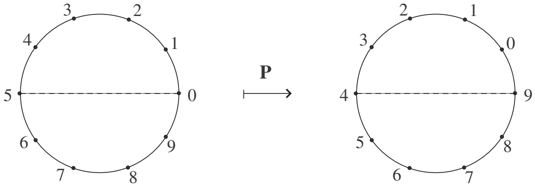

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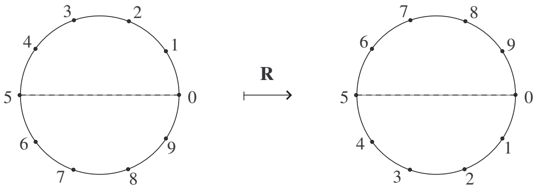

Circular Reversal $\mathbf{R}$

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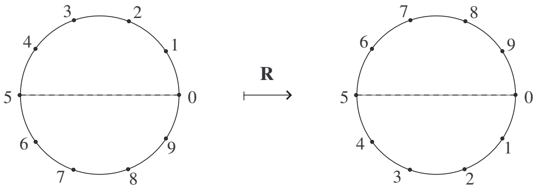

- $$\mathbf{R} = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \swarrow & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 \end{bmatrix}$$

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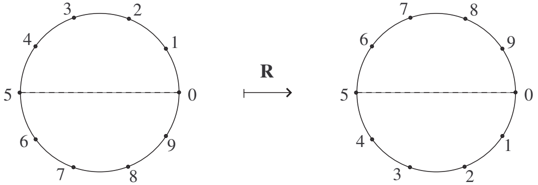

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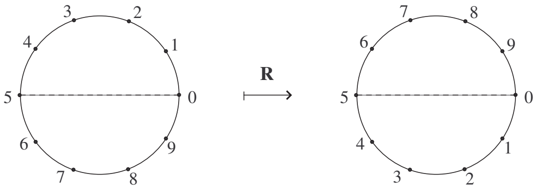

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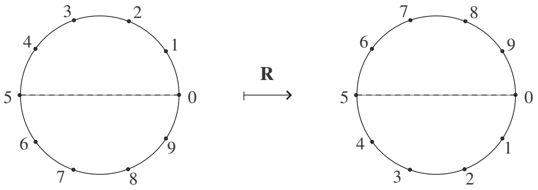

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  - $\mathbf{x}^{(3)} = \begin{bmatrix} 0 & 6 & 6 & 6 & 6 & 6 \end{bmatrix}^T$
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  - $\mathbf{x}^{(5)} = \begin{bmatrix} 0 & -1 & 2 & -3 & 4 & -5 \end{bmatrix}^T$

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  - $\mathbf{x}^{(5)} = \left[\begin{array}{cccccc} 0 & -1 & 2 & -3 & 4 & -5 \end{array}\right]^T$
  - $\mathbf{x}^{(6)} = \left[\begin{array}{cccccc} 0 & 2 & 0 & 6 & 0 & 10 \end{array}\right]^T$

Some Identities

- $\mathbf{W} \; = \; \mathbf{V}^* \; = \; \mathbf{VR} \; = \; \mathbf{RV}$
- $\mathbf{V} \; = \; \mathbf{W}^* \; = \; \mathbf{WR} \; = \; \mathbf{RW}$