

Lecture 9

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- ▶ Some issues in solving $\mathbf{Ax} = \mathbf{b}$

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- ▶ Nonsingularity and the matrix inverse \mathbf{A}^{-1}
- ▶ Properties of \mathbf{A}^{-1}
- ▶ Inversion of a triangular matrix

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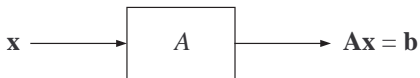
- ▶ Some issues in solving $\mathbf{Ax} = \mathbf{b}$
- ▶ Nonsingularity and the matrix inverse \mathbf{A}^{-1}
- ▶ Properties of \mathbf{A}^{-1}
- ▶ Inversion of a triangular matrix
- ▶ Solution of $\mathbf{Ax} = \mathbf{b}$ by Gaussian elimination

Solving $\mathbf{Ax} = \mathbf{b}$

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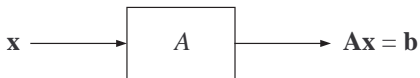


Solving $\mathbf{Ax} = \mathbf{b}$



- ▶ Problem: determine the input \mathbf{x} of a linear transformation (or system) based on the observed output $\mathbf{y} = \mathbf{b}$

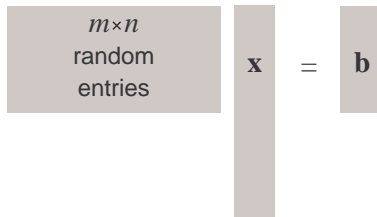
Solving $\mathbf{Ax} = \mathbf{b}$



- ▶ Problem: determine the input \mathbf{x} of a linear transformation (or system) based on the observed output $\mathbf{y} = \mathbf{b}$
- ▶ The dimensions of the input (n) and output (m) play a crucial role here.

The Case $m < n$

The Case $m < n$



The diagram illustrates a linear system $Ax = b$ where A is an $m \times n$ matrix of random entries, x is a column vector, and b is a column vector. The matrix A is represented by a gray rectangular box containing the text " $m \times n$ random entries". The vector x is represented by a tall, narrow gray vertical bar. The vector b is represented by a shorter, wider gray vertical bar. The equation is shown as $A = x = b$.

The Case $m < n$

The diagram illustrates the equation $Ax = b$. On the left, a gray rectangular box contains the text " $m \times n$ random entries". To its right is a vertical gray bar labeled x . Further right is an equals sign, followed by another vertical gray bar labeled b .

- ▶ Every b is almost certainly a valid system output, thus a solution x exists.

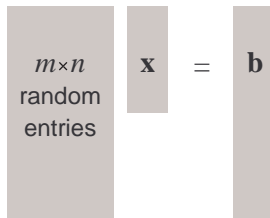
The Case $m < n$

A diagram illustrating a linear system. On the left, a gray rectangular box contains the text " $m \times n$ random entries". To its right is a vertical gray bar labeled \mathbf{x} . Further right is an equals sign, followed by another vertical gray bar labeled \mathbf{b} .

- ▶ Every \mathbf{b} is almost certainly a valid system output, thus a solution \mathbf{x} exists.
- ▶ The solution is *not* unique, thus the true input cannot be determined.

The Case $m > n$

The Case $m > n$



The diagram illustrates a linear system $Ax = b$ where the matrix A is $m \times n$ with $m > n$. The matrix A is represented by a tall, narrow gray rectangle containing the text " $m \times n$ random entries". To its right is a shorter, narrower gray rectangle labeled x . An equals sign follows, and to its right is another tall, narrow gray rectangle labeled b . The height of the b vector is the same as the height of the A matrix, indicating m equations. The height of the x vector is shorter, indicating n unknowns.

$$\begin{matrix} m \times n \\ \text{random} \\ \text{entries} \end{matrix} \quad \mathbf{x} \quad = \quad \mathbf{b}$$

The Case $m > n$

A diagram illustrating a linear system $Ax = b$. On the left is a tall, light gray rectangular block representing the matrix A , with the text " $m \times n$ random entries" inside. To its right is a shorter, light gray rectangular block representing the vector x . To the right of x is an equals sign, followed by another tall, light gray rectangular block representing the vector b . The height of A is greater than the height of x , and the height of b is equal to the height of A .

- ▶ A *random* b is almost certainly *not* a valid system output, thus a solution does not exist.

The Case $m > n$

A diagram illustrating a linear system $Ax = b$. On the left is a tall, narrow gray rectangle representing the matrix A , with the text " $m \times n$ random entries" inside. To its right is a shorter, narrower gray rectangle representing the vector x . To the right of x is an equals sign, followed by another tall, narrow gray rectangle representing the vector b . The height of A is greater than the height of x , and the height of b is equal to the height of x .

- ▶ A *random* b is almost certainly *not* a valid system output, thus a solution does not exist.
- ▶ If b is a valid system output, a solution x exists and is almost certainly unique.

The Case $m = n$

The Case $m = n$

The diagram illustrates the equation $A\mathbf{x} = \mathbf{b}$. On the left, a large gray square represents the matrix A , containing the text " $m \times n$ random entries". To its right is a narrow gray vertical bar representing the vector \mathbf{x} . An equals sign "=" is placed between the vector \mathbf{x} and another narrow gray vertical bar representing the vector \mathbf{b} .

The Case $m = n$

A diagram illustrating a linear system. On the left is a large gray square containing the text " $m \times n$ random entries". To its right is a vertical gray bar labeled \mathbf{x} . Further right is an equals sign, followed by another vertical gray bar labeled \mathbf{b} .

- ▶ Every \mathbf{b} is almost certainly a valid system output corresponding to a unique system input \mathbf{x} .

The Case $m = n$

A diagram illustrating the equation $A\mathbf{x} = \mathbf{b}$. On the left is a large gray square representing the matrix A , containing the text " $m \times n$ random entries". To its right is a narrow gray vertical bar representing the column vector \mathbf{x} . An equals sign "=" is placed between the vector \mathbf{x} and another narrow gray vertical bar representing the column vector \mathbf{b} .

- ▶ Every \mathbf{b} is almost certainly a valid system output corresponding to a unique system input \mathbf{x} .
- ▶ In other words, a solution \mathbf{x} exists for every \mathbf{b} , and is unique.

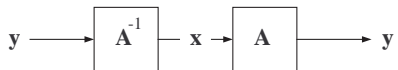
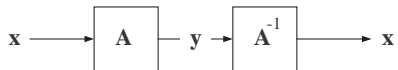
Properties of \mathbf{A}^{-1}

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▶ $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

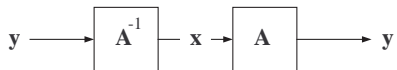
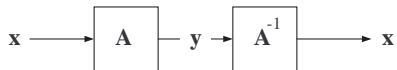
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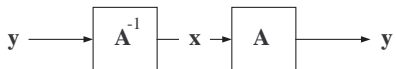
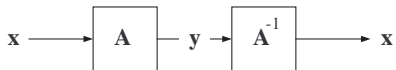
▶ $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$



▶ $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Properties of \mathbf{A}^{-1}

▶ $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$



▶ $(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$



Example: Gaussian Elimination

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$$2x_1 + x_2 - x_3 = 6$$

$$4x_1 - x_3 = 6$$

$$-8x_1 + 2x_2 + 3x_3 = -10$$

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