

LECTURE 1

Topic: complex numbers

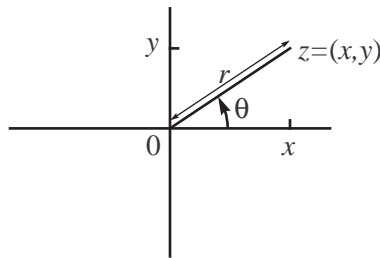
Textbook References: Section 1.3

Key Points:

- A complex number z is a point on a two-dimensional plane (the complex plane). It can be specified using either Cartesian (x, y) or polar (r, θ) coordinates.
- Addition and scaling of complex numbers follows the same rules as for (two-dimensional) vectors.

Theory and Examples:

1. A complex number z is a point (or vector) on a two-dimensional plane, known as the complex plane and represented by \mathbf{C} .



The Cartesian coordinates of z are

$$x = \Re\{z\}, \text{ the real part of } z$$

$$y = \Im\{z\}, \text{ the imaginary part of } z$$

and the corresponding axes are known as the real and imaginary axes, respectively.

The polar coordinates of z are

$$r = |z|, \text{ the modulus, or magnitude, of } z$$

$$\theta = \angle z, \text{ the angle of } z$$

2. The usual rules for converting between coordinate systems apply:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r = \sqrt{x^2 + y^2}$$

As for the angle θ , it is customary to quote it in radians. Note that $2\pi \text{ rad} = 360^\circ =$ one full revolution, and thus angles which differ by multiples of 2π are equivalent. Usually, θ in

quoted in the interval $[0, 2\pi)$ or $(-\pi, \pi]$. We can obtain θ from (x, y) by careful use of the arctan function:

$$\theta = \arctan\left(\frac{y}{x}\right) + (0 \text{ or } \pi)$$

where π is added if and only if x is negative.

- 3. Example.** Consider the complex numbers z_1 and z_2 , where

$$\Re\{z_1\} = -5, \quad \Im\{z_1\} = 2$$

and

$$|z_2| = 4, \quad \angle z_2 = \pi/6 \text{ rad}$$

Your task: Plot these on the complex plane.

The modulus of z_1 is given by

$$|z_1| = \sqrt{(-5)^2 + 2^2} = 5.3852$$

and its angle by

$$\angle z_1 = \arctan(-2/5) + \pi = 2.7611 \text{ rad}$$

or 158.2° . Note that π was added to the result of $\arctan(\cdot)$ because the real part of z_1 is negative.

As for z_2 , we have

$$\begin{aligned} \Re\{z_2\} &= 4 \cos(\pi/6) = 2\sqrt{3} = 3.4641 \\ \Im\{z_2\} &= 4 \sin(\pi/6) = 2 \end{aligned}$$

Your task: Using as little algebra as possible, repeat for

$$\Re\{z_3\} = 5, \quad \Im\{z_3\} = -2$$

and

$$|z_4| = 4, \quad \angle z_4 = -\pi/6 \text{ rad}$$

- 4.** Since complex numbers are vectors, expressions such as cz (scaling by a real constant c) or $z_1 + z_2$ (summation) have the same meaning as in the case of two-dimensional vectors. Clearly, summation of two complex numbers is easiest to perform using Cartesian coordinates (i.e., real and imaginary parts).

LECTURE 2

Topic: lines and circles on the complex plane; complex multiplication and division

Textbook References: section 1.3

Key Points:

- Two complex numbers can be multiplied by expressing each number in the form $z = x + jy$, then using distributivity and the rule $j^2 = -1$ (i.e., j is treated as the square root of -1).
- The product of two complex numbers with polar coordinates (r_1, θ_1) and (r_2, θ_2) is the complex number with polar coordinates $(r_1 r_2, \theta_1 + \theta_2)$.
- The product of a complex number z and its conjugate z^* equals the square modulus $|z|^2$.
- If z has polar coordinates (r, θ) , its inverse z^{-1} has polar coordinates $(r^{-1}, -\theta)$.

Theory and Examples:

1. The modulus $|z|$ represents the length of the vector z . Similarly, the expression $|z_1 - z_2|$ gives the distance between the points z_1 and z_2 . Using this concept, we see that the following two equations in the variable z each have a familiar geometrical interpretation:
 - $|z - z_0| = a$, where a is a positive constant, is the set of points z on the complex plane which are at a fixed distance a from the point z_0 . This is a circle of radius a centered at z_0 .
 - $|z - z_1| = |z - z_2|$ is the set of points z on the complex plane which are equidistant from the points z_1 and z_2 . This is the same as the perpendicular bisector of the line segment joining z_1 and z_2 .

Your task: Sketch the two curves given by the equations

$$|z - 3 + 4j| = 5$$

and

$$|z - 3| = |z + 4j|$$

2. The most common form for a complex number z incorporates the real and imaginary parts as follows:

$$z = x + jy$$

This form, together with the convention that $j \times j = j^2 = -1$, allows us to multiply two complex numbers together. For example,

$$\begin{aligned}(5 - 2j)(3 - 4j) &= 15 - 20j - 6j + 8j^2 \\ &= 7 - 26j\end{aligned}$$

3. In polar form, multiplication of complex numbers is simple. If

$$z_1 = r_1(\cos \theta_1 + j \sin \theta_1) \quad \text{and} \quad z_2 = r_2(\cos \theta_2 + j \sin \theta_2)$$

then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos \theta_1 + j \sin \theta_1)(\cos \theta_2 + j \sin \theta_2) \\ &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + j(\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2)] \end{aligned}$$

But

$$\begin{aligned} \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 &= \cos(\theta_1 + \theta_2) \\ \cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2 &= \sin(\theta_1 + \theta_2) \end{aligned}$$

and thus $z = z_1 z_2$ has modulus equal to the product of the two moduli, and angle equal to the sum of the two angles.

4. **Example.** As before, take

$$z_1 = 5 - 2j \quad \text{and} \quad z_2 = 3 - 4j$$

The polar forms are

$$|z_1| = \sqrt{29}, \quad \angle z_1 = -0.3805$$

and

$$|z_2| = 5, \quad \angle z_2 = -0.9273$$

(The angles were obtained using the MATLAB `ANGLE` function, which returns values between $-\pi$ and π .) Therefore

$$|z_1 z_2| = 5\sqrt{29}, \quad \angle z_1 z_2 = -0.3805 - 0.9273 = -1.3078$$

and, in Cartesian form

$$\begin{aligned} z_1 z_2 &= 5\sqrt{29} \cdot (\cos(-1.3078) + j \sin(-1.3078)) \\ &= 7 - j26 \end{aligned}$$

Extension: Powers of z_1 and z_2 , and products thereof, can be also computed easily in polar form. Thus

$$\begin{aligned} |z_1^5 z_2^2| &= 29^{5/2} \cdot 5^2 = 1.1322 \times 10^5, \\ \angle z_1^5 z_2^2 &= 5 \cdot (-0.3805) + 2 \cdot (-0.9273) = -3.7571 \end{aligned}$$

and

$$\begin{aligned} z_1^5 z_2^2 &= (1.1322 \times 10^5) \cdot (\cos(-3.7571) + j \sin(-3.7571)) \\ &\approx 92442 + j65370 \end{aligned}$$

where the real and imaginary parts were rounded to the nearest integers. The calculation was deliberately carried out using low precision (four or five digits) in the intermediate results, in order to illustrate the accumulation of roundoff errors. The correct result is $92443 + j65374$.

5. Division of two complex numbers makes implicit use of the inverse, i.e.,

$$z_1/z_2 = z_1(1/z_2) = z_1z_2^{-1}$$

This can be carried out in a single step by multiplying both the numerator z_1 and the denominator z_2 by the complex conjugate of z_2 (which differs from z_2 in the sign of the imaginary part). Thus If $z_1 = 5 - 2j$ and $z_2 = 3 - 4j$, then

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{5 - 2j}{3 - 4j} \\ &= \frac{(5 - 2j)(3 + 4j)}{(3 - 4j)(3 + 4j)} \\ &= \frac{(5 - 2j)(3 + 4j)}{3^2 + 4^2} \\ &= \frac{23}{25} + j\frac{14}{25}\end{aligned}$$

Here we had an instance of the identity

$$zz^* = |z|^2$$

i.e., the product of a complex number and its conjugate equals the (real-valued) square magnitude of that number.

6. In polar form, dividing two complex numbers corresponds to dividing their magnitudes and subtracting one angle from the other. This is because

$$z^{-1} = z^*/|z|^2$$

and therefore

$$|z^{-1}| = |z|/|z|^2 = |z|^{-1}, \quad \angle z^{-1} = \angle z^* = -\angle z$$

Your task: Verify the above results. Plot z , z^* and z^{-1} for $z = 3 - 4j$.

LECTURE 3

Topic: complex exponentials; n^{th} root of a complex number; continuous-time sinusoids

Textbook References: sections 1.3 and 1.4

Key Points:

- The generic complex number $z = r(\cos \theta + j \sin \theta)$ can be also written as $z = re^{j\theta}$.
- As is the case with real exponentials, $e^{j(\theta+\phi)} = e^{j\theta} e^{j\phi}$
- The functions $\cos \theta$ and $\sin \theta$ can be expressed in terms of complex exponentials:

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

- The equation $z^n = e^{j\phi}$, where ϕ is a given angle, has n roots of the form $z = e^{j\theta}$; these are obtained by setting $\theta = (\phi + 2k\pi)/n$, where $k = 0, \dots, n - 1$.
- The functions $\cos \theta$ and $\sin \theta$ are both periodic with period 2π (radians), and are shifted versions of each other.
- The generic time-dependent sinusoid $A \cos(\Omega t + \phi)$ has three parameters: amplitude A , angular frequency Ω (rad/sec) and initial phase ϕ . The cyclic frequency f (Hz) and period T (sec) are related to Ω by

$$f = 1/T = \frac{\Omega}{2\pi}$$

Theory and Examples:

1. Take a complex number z with modulus $|z| = r$ and angle $\angle z = \theta$:

$$z = r(\cos \theta + j \sin \theta)$$

An alternative form for z is

$$z = re^{j\theta}$$

where we use the identity

$$e^{j\theta} = \cos \theta + j \sin \theta$$

This can be obtained from the Taylor series

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$

by setting $t = j\theta$. Grouping the real and imaginary terms separately on the expansion side, we recognize the two Taylor series

$$\begin{aligned} \cos \theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \quad (\text{real part}) \\ \sin \theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \quad (\text{imaginary part}) \end{aligned}$$

2. **Example.** In polar form, $z_1 = -1 + j$ and $z_2 = 1 + j\sqrt{3}$ are given by

$$|z_1| = \sqrt{2}, \quad \angle z_1 = 3\pi/4$$

and

$$|z_2| = 2, \quad \angle z_2 = \pi/3$$

In complex exponential form,

$$z_1 = \sqrt{2}e^{j(3\pi/4)} \quad \text{and} \quad z_2 = 2e^{j(\pi/3)}$$

3. The multiplication/division rules for the polar form can be easily explained using the identity

$$e^{a+b} = e^a \cdot e^b$$

Thus

$$r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1 + \theta_2)} \quad \text{and} \quad \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} \cdot e^{j(\theta_1 - \theta_2)}$$

Question: If $z = re^{j\theta}$, what is the complex exponential form of z^* ?

4. By adding/subtracting the equations

$$\begin{aligned} e^{j\theta} &= \cos \theta + j \sin \theta \\ e^{-j\theta} &= \cos(-\theta) + j \sin(-\theta) = \cos \theta - j \sin \theta \end{aligned}$$

we get the following relationships for the sine and cosine functions:

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

5. The equation

$$z^n = v,$$

where z is a complex variable and v is a complex constant, has n complex roots (this is true for any polynomial of degree n). These roots can be found using the following observations:

- $|z|^n = |v|$, therefore every root z must satisfy $|z| = |v|^{1/n}$, i.e., it must lie on a circle of radius $|v|^{1/n}$ centered at the origin;
- on that circle, the complex number z with

$$\angle z = (\angle v)/n$$

is a root;

- on the same circle, the angles

$$(\angle v)/n + k(2\pi/n), \quad k = 1, \dots, n-1$$

also correspond to roots (the remaining $n-1$ roots of the equation).

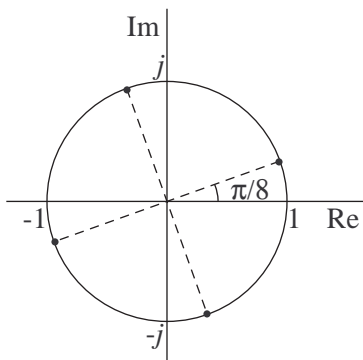
6. **Example.** To determine the four roots of

$$z^4 = j,$$

we note that $j = e^{j\pi/2}$. Since $|j| = 1$, all the roots of the equation lie on the unit circle. And since $\angle j = \pi/2$, the roots are

$$e^{j(\pi/8)}, \quad e^{j(5\pi/8)}, \quad e^{j(9\pi/8)} \quad \text{and} \quad e^{j(13\pi/8)}$$

Your task: Determine and sketch the roots of $z^3 = -8$.



7. The functions $\cos \theta$ and $\sin \theta$ (where θ is in radians) are both periodic with period 2π :

$$\begin{aligned} \cos(\theta + 2\pi) &= \cos \theta \\ \sin(\theta + 2\pi) &= \sin \theta \end{aligned}$$

The cosine has (even) symmetry about $\theta = 0$, the sine has odd symmetry (or antisymmetry):

$$\begin{aligned} \cos(-\theta) &= \cos \theta \\ \sin(-\theta) &= -\sin \theta \end{aligned}$$

Either function can be obtained from the other by shifting θ by $\pi/2$ in the appropriate direction:

$$\begin{aligned} \sin \theta &= \cos(\theta - \pi/2) \\ \cos \theta &= \sin(\theta + \pi/2) \end{aligned}$$

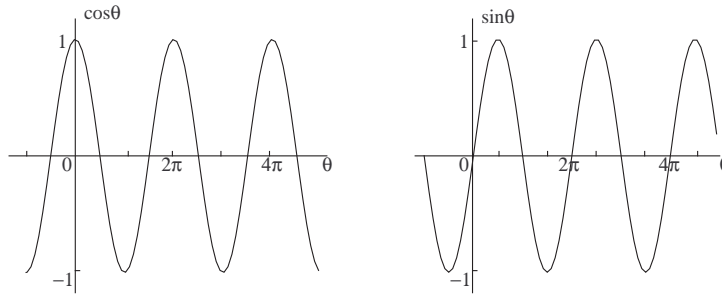
A shift in θ by π (same as $-\pi$) results in sign reversal in each case:

$$\begin{aligned} \cos(\theta + \pi) &= -\cos \theta \\ \sin(\theta + \pi) &= -\sin \theta \end{aligned}$$

8. The generic continuous-time sinusoid is given by

$$x(t) = A \cos(\Omega t + \phi)$$

where



- $A > 0$ is the amplitude
- Ω is the angular frequency, in rad/sec
- t is in seconds
- ϕ is the initial phase, in radians

Since $\cos \theta$ has period 2π radians, $x(t)$ has period $T = 2\pi/\Omega$ seconds. The cyclic frequency (i.e., periods or cycles per second) is given by

$$f = 1/T = \frac{\Omega}{2\pi}$$

and is expressed in $\text{Hz} = 1/\text{sec} = 1 \text{ cycle/sec}$.

9. Example. Suppose that you are given the following information about a continuous-time sinusoid $x(t) = A \cos(\Omega t + \phi)$:

- $x(t) \geq 1.50$ for 23.0% of its period;
- it takes 0.02 seconds for the value of the sinusoid to drop from 1.50 to 0.00; and
- $x(0) = -0.3$ and the first derivative $x'(0)$ is positive.

You have three (and then some) independent pieces of information and three unknowns (A , Ω and ϕ), so you should be able to solve for these parameters.

Justify each of the following results:

- The first piece of information gives $A = 1.5/\cos(0.23\pi) = 2.0$.
- In addition, the second piece of information gives

$$\frac{0.5\pi - 0.23\pi}{\Omega} = 0.02 \quad \Rightarrow \quad \Omega = 13.5\pi$$

Question: What are the values of f (in Hz) and T (in sec)?

- With A known, the third piece of information gives $\phi = -\cos^{-1}(-0.3/2.0) = -1.721$ rad.

LECTURE 4

Topic: phasors; discrete-time sinusoids

Textbook References: sections 1.4, 1.5

Key Points:

- The stationary phasor of $A \cos(\Omega t + \phi)$ is the complex number $Ae^{j\phi}$. The sum of two (or more) sinusoids of arbitrary amplitudes and phases but of identical frequency Ω is a sinusoid of frequency Ω . Sinusoids of identical frequency can be added together by taking the complex sum of their stationary phasors.
- The discrete time parameter n counts samples. The (angular) frequency parameter ω is an angle increment (radians/sample). Physical time (seconds) is nowhere involved.
- Frequencies ω and $\omega + 2\pi$ are equivalent (i.e., produce the same signal) for real or complex sinusoids in discrete time.
- Frequencies ω and $2\pi - \omega$ can be used alternatively to describe a real sinusoid in discrete time:

$$\cos(\omega n + \phi) = \cos(-\omega n - \phi) = \cos((2\pi - \omega)n - \phi)$$

- The effective range of frequencies for a real sinusoid in discrete time is 0 (lowest) to π (highest).
- A discrete-time sinusoid is periodic if and only if ω is of the form

$$\omega = \frac{k}{N} \cdot 2\pi$$

for integers k and N . The fundamental period is the smallest value of N for which the above holds.

Theory and Examples:

1. Since $\cos \theta = \Re\{e^{j\theta}\}$, it follows that $x(t)$ is the real part of the time-dependent complex sinusoid

$$z(t) = Ae^{j(\Omega t + \phi)}$$

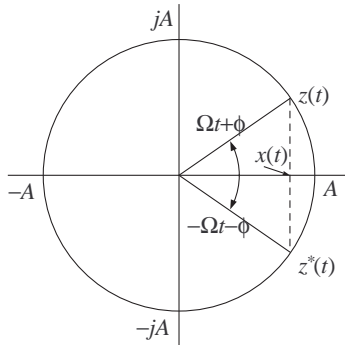
On the complex plane, the point $z(t)$ moves with constant angular velocity Ω on a circle of radius A . Its projection on the real axis equals $x(t)$. The initial position

$$z(0) = Ae^{j\phi},$$

viewed as a vector, is known as the stationary phasor of $x(t)$.

2. Two real-valued sinusoids of the same frequency can be added together:

$$\begin{aligned} A_1 \cos(\Omega t + \phi_1) + A_2 \cos(\Omega t + \phi_2) &= \Re \left\{ A_1 e^{j(\Omega t + \phi_1)} + A_2 e^{j(\Omega t + \phi_2)} \right\} \\ &= \Re \left\{ (A_1 e^{j\phi_1} + A_2 e^{j\phi_2}) e^{j\Omega t} \right\} \\ &= A \cos(\Omega t + \phi) \end{aligned}$$



where

$$Ae^{j\phi} = A_1e^{j\phi_1} + A_2e^{j\phi_2}$$

The result is a sinusoid of the same frequency, whose stationary phasor is the complex (i.e., vector) sum of two component stationary phasors.

3. Example.

$$2.7 \cos(15\pi t + 0.6) + 4.1 \sin(15\pi t - 1.8) = A \cos(15\pi t + \phi)$$

where

$$Ae^{j\phi} = 2.7e^{j0.6} + 4.1e^{j(-1.8-\pi/2)}$$

We convert each term to its Cartesian form, compute the sum and convert back to polar form to obtain $A = 3.0241$ and $\phi = 2.1937$.

Your task: Fill in the missing steps. What happened to $\sin(\cdot)$?

4. A discrete-time signal is a sequence of values (samples) $x[n]$, where n ranges over all integers. A discrete-time sinusoid has the general form

$$x[n] = A \cos(\omega n + \phi)$$

or, in its complex version,

$$z[n] = Ae^{j(\omega n + \phi)}$$

Question: How is $x[n]$ related to $z[n]$?

5. **Your task:** Use MATLAB to generate 100 values of each of the discrete-time sinusoids $x_1[n]$ and $x_2[n]$:

```
n = 0:99;
w1 = pi/25; q1 = 2*pi/5;
x1 = cos(w1*n + q1);
w2 = 2.4; q2 = -1.3;
x2 = cos(w2*n + q2);
bar(n,x1)           % discrete bar graph
plot(n,x1), grid   % extrapolated graph
bar(n,x2)           % no resemblance to a continuous-time sinusoid
```

Depending on its frequency, a discrete-time sinusoid may look similar to, or quite different from, a continuous-time one.

6. The frequency parameter ω is measured in radians, or radians per sample. (Unlike Ω , which is in radians per second). Thus the frequency of a discrete-time sinusoid is just an angle increment: the argument of $\cos(\cdot)$ increases by a fixed amount ω with each sample.

Two key observations:

- ω and $\omega + 2k\pi$, where k is an integer, represent the same frequency. This is because ωn and $(\omega + 2k\pi)n$ differ by $2kn\pi$ radians, i.e., a whole number of revolutions, and therefore at every time n ,

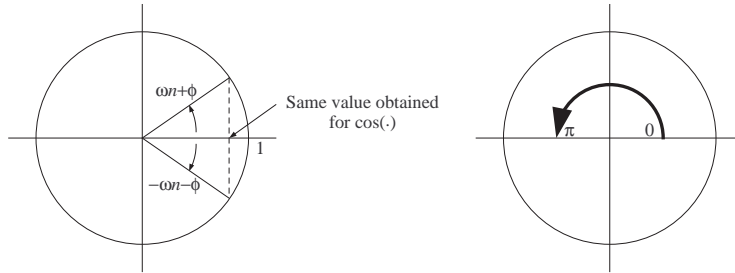
$$\begin{aligned}\cos(\omega n + \phi) &= \cos((\omega + 2k\pi)n + \phi) \\ e^{j(\omega n + \phi)} &= e^{j((\omega + 2k\pi)n + \phi)}\end{aligned}$$

Typically, the range of ω is chosen as $[0, 2\pi)$ or $(-\pi, \pi]$.

- In the real-valued case, either ω or $-\omega$ can be used to express the same sinusoid. This is due to the identity $\cos \theta = \cos(-\theta)$, which implies that for every n ,

$$\cos(\omega n + \phi) = \cos(-\omega n - \phi)$$

As a result, the range of ω for real-valued sinusoids can be limited to $[0, \pi]$.



7. **Example.** We are asked to express

$$z[n] = e^{j((37\pi/13)n + \pi/8)} \quad \text{and} \quad x[n] = \cos((42\pi/13)n - 5\pi/6)$$

using the smallest positive equivalent frequency ω in each case. Note that both frequencies ($37\pi/13$ and $42\pi/13$) are greater than 2π , so we need to subtract multiples of 2π . For the complex-valued sinusoid, the answer is

$$z[n] = e^{j((11\pi/13)n + \pi/8)}$$

For the real-valued sinusoid, we have

$$\frac{42\pi}{13} - 2\pi = \frac{16\pi}{13} \quad \text{and} \quad \frac{42\pi}{13} - 4\pi = -\frac{10\pi}{13}$$

Therefore the answer is

$$x[n] = \cos((10\pi/13)n + 5\pi/6)$$

- 8. Your task:** Find simple expressions for $x[n] = A \cos(\omega n + \phi)$ when $\omega = 0$ (lowest possible frequency) and $\omega = \pi$ (highest possible frequency). **Also:** Modify the MATLAB script given earlier to compute and plot 100 values of the high-frequency sinusoid

$$x_3[n] = \cos((24\pi/25)n + 2\pi/5)$$

Note that the frequencies of $x_1[n]$ and $x_3[n]$ are complementary to each other in the interval $[0, \pi]$.

- 9.** The fundamental period of $x[n]$ is the smallest integer N such that

$$(\forall n) \quad x[n + N] = x[n]$$

If no such N exists, then the signal is nonperiodic (or aperiodic).

The sequences $\cos(\omega n + \phi)$ and $e^{j(\omega n + \phi)}$ are repetitions of a fixed vector of N values if and only if the argument $\omega n + \phi$ changes by an exact multiple of 2π every N time indices. In other words, if and only if

$$\omega N = 2k\pi \quad \Leftrightarrow \quad \omega = \frac{k}{N} \cdot 2\pi$$

for some integer k . The smallest value of N satisfying the above relationship is the fundamental period.

- 10. Example.** Shown is the fundamental period N (where periodic).

$$\begin{aligned} \omega = 0 &\Rightarrow N = 1 \\ \omega = \pi &\Rightarrow N = 2 \\ \omega = 1.0 &\Rightarrow N = \infty \quad (\text{i.e., nonperiodic}) \\ \omega = 10\pi/13 &\Rightarrow N = 13 \\ \omega = 11\pi/13 &\Rightarrow N = 26 \end{aligned}$$

Note that for a periodic discrete-time sinusoid, the fundamental period does not necessarily equal $2\pi/\omega$ (as was the case with continuous-time sinusoids).

LECTURE 5

Topic: sampling of continuous-time sinusoids

Textbook References: section 1.6

Key Points:

- Sampling a continuous-time sinusoid at a rate of $f_s = 1/T_s$ (samples/second) produces a discrete-time sinusoid.
- If the continuous-time sinusoid has angular frequency $\Omega = 2\pi f = 2\pi/T$, the resulting discrete-time sinusoid has angular frequency

$$\omega = \Omega T_s = 2\pi \cdot \frac{f}{f_s} = 2\pi \cdot \frac{T_s}{T}$$

- At high sampling rates, discrete-time samples capture the variation of the continuous-time signal in great detail.
- Two different sampling rates $f_s = 1/T_s$ and $f'_s = 1/T'_s$ will produce samples having the same effective frequency provided the sum $T_s + T'_s$ or the difference $T_s - T'_s$ is an integer multiple of $T = 1/f$.

Theory and Examples:

1. The sampling formula

$$x[n] = x(nT_s)$$

produces a sequence of samples $x[n]$ from a continuous-time signal $x(t)$. T_s is the sampling period and $1/T_s$ is the sampling rate. Applied to the continuous-time sinusoid

$$x(t) = \cos(\Omega t + \phi) ,$$

this formula produces the discrete-time sinusoid

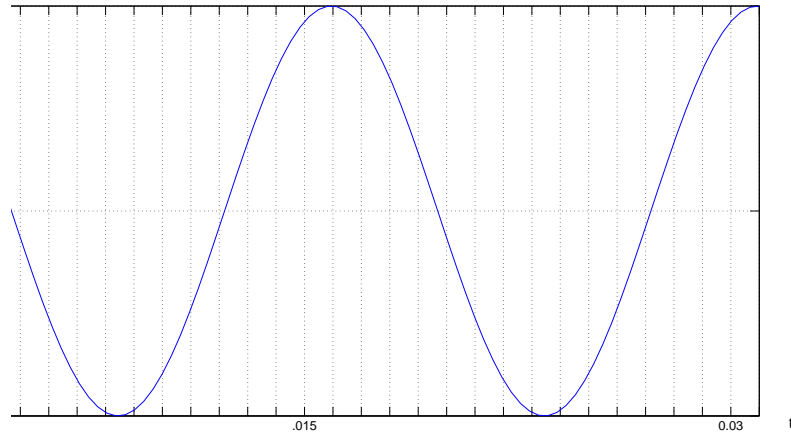
$$x[n] = \cos(\Omega T_s n + \phi)$$

2. **Example.** The graph below shows a segment of the sinusoid

$$x(t) = \cos((400\pi/3)t - \pi/8)$$

Your task: Using two types of stem lines, mark the samples in each of

$$\begin{aligned} x_1[n] &= \cos((2\pi/5)n - \pi/8) && \text{(i.e., } T_s = 3.0 \text{ ms)} \\ x_2[n] &= \cos((2\pi/3)n - \pi/8) && \text{(i.e., } T_s = 5.0 \text{ ms)} \end{aligned}$$



3. The frequency of the discrete-time sinusoid obtained by sampling is

$$\omega = \Omega T_s = 2\pi \cdot \frac{f}{f_s} = 2\pi \cdot \frac{T_s}{T}$$

Using this equation, it is easy to determine the relationship between sampling rate f_s (samples/sec) to the frequency f (Hz) in order for the discrete-time sinusoid to be periodic.

Questions:

- What is this relationship?
 - What is the equivalent condition in terms of T_s and T ?
 - For what values of T_s is the discrete-time sequence $x[n]$ constant in n ?
 - For what values of T_s is $x[\cdot]$ of the form $\dots, a, -a, a, -a, \dots$?
4. If the ratio T_s/T is small (i.e., f_s/f is large), the sequence of samples $x[n]$ closely tracks the variation in $x(t)$. The smaller the ratio T_s/T , the easier it becomes to interpolate the discrete samples so as to approximate to the continuous-time signal. In effect, this is what MATLAB does in producing a continuous plot.
5. The range of values

$$T_s \in [0, T/2] \quad \text{or equivalently} \quad f_s \in [2f, \infty]$$

corresponds to $\omega \in [0, \pi]$, which is the effective frequency range for real sinusoids. This means that if $T_s > T/2$, or equivalently, if $f_s < 2f$, the sequence of samples obtained from $x(t)$ has the same frequency as one obtained using a smaller sampling period (or higher sampling rate). There are, in fact, infinitely many sampling rates that produce effectively the same ω in $x[n]$. To see this, recall that ω and ω' can be used to describe the same discrete-time sinusoid provided

$$\omega' = \pm\omega + 2k\pi$$

for some integer k . Thus if f_s and f'_s are two sampling rates such that

$$\frac{f}{f_s} \pm \frac{f}{f'_s} = k$$

or equivalently, if

$$T_s \pm T'_s = kT ,$$

then the two resulting sample sequences will have the same (angular) frequency, provided that frequency is expressed in the interval $[0, \pi]$.

6. Example. Consider the continuous-time sinusoid

$$x(t) = \cos((400\pi/3)t - \pi/8)$$

and the discrete-time sinusoids

$$\begin{aligned} x_1[n] &= \cos((2\pi/5)n - \pi/8) && \text{(as before)} \\ x_3[n] &= \cos((2\pi/5)n + \pi/8) \end{aligned}$$

Here $T = 15.0$ ms, $f = 200/3$ Hz.

- The smallest T_s which produces $x_1[\cdot]$ equals (from $\omega = \Omega T_s$)

$$T_s = \frac{2\pi}{5} \cdot \frac{3}{400\pi} = 3.0 \text{ ms}$$

The remaining values of T_s which produce $x_1[\cdot]$ are given by

$$T_s = 3.0 + k(15.0) \text{ ms}, \quad k = 1, 2, \dots$$

- As for $x_3[\cdot]$, we use $\cos(-\theta) = \cos(\theta)$ to express it as

$$x_3[n] = \cos(-(2\pi/5)n - \pi/8)$$

The values of T_s which produce $x_3[\cdot]$ are thus

$$T_s = -3.0 + k(15.0) \text{ ms}, \quad k = 0, 1, \dots$$

Since T_s is necessarily positive, it follows that

$$T_s = 12.0 + k(15.0) \text{ ms}, \quad k = 1, 2, \dots$$

LECTURE 6

Topic: aliasing

Textbook References: section 1.6

Key Points:

- Two continuous-time sinusoids having different frequencies f and f' (Hz) may, when sampled at the same sampling rate f_s , produce sample sequences having effectively the same frequency. This phenomenon is known as aliasing, and occurs when

$$f \pm f' = kf_s$$

for some integer k .

- If a continuous-time signal consisting of additive sinusoidal components is sampled uniformly, reconstruction of that signal from its samples is impossible if aliasing has occurred between any two components at different frequencies.
- If the sinusoidal components of a continuous-time signal span the frequency range 0 to f_B (Hz), aliasing is avoided if and only if the sampling rate f_s exceeds $2f_B$, a figure known as the Nyquist rate.

Theory and Examples:

1. We saw that a continuous-time sinusoid of frequency $f = 1/T$ (Hz) can be sampled at two different rates $f_s = 1/T_s$ and $f'_s = 1/T'_s$ to produce sample sequences having the same effective frequency. This happens whenever

$$T_s \pm T'_s = kT$$

for some integer k .

2. An analogous phenomenon occurs when two continuous-time sinusoids having different frequencies f and f' are sampled at the same rate f_s : depending on the value of f_s , the two sample sequences may have the same effective frequency. When this happens, we say that f and f' are aliases (of each other) with respect to the sampling rate f_s . In mathematical terms, we know that

$$\omega = 2\pi \cdot \frac{f}{f_s} \quad \text{and} \quad \omega' = 2\pi \cdot \frac{f'}{f_s}$$

can be used to represent the same discrete-time sinusoid provided

$$\omega' = \pm\omega + 2k\pi$$

for some integer k . Thus f and f' are aliases with respect to f_s provided

$$f \pm f' = kf_s$$

3. **Example.** Let $f = 150$ Hz and $f_s = 400$ samples/sec. Then the aliases of f are given (in Hz) by

$$f' = 150 + k(400) \quad \text{and} \quad f' = -150 + k(400)$$

If we reduce f_s to 280 samples/sec, then the aliases of f are given by

$$f' = 150 + k(280) \quad \text{and} \quad f' = -150 + k(280)$$

Your task: In each case ($f_s = 400$ and $f_s = 280$), determine all the aliases in the range 0 to 2,000 Hz.

4. **Example.** Suppose that $x(t)$ is a sinusoid whose frequency is between 600 and 800 Hz. It is sampled at a rate $f_s = 400$ samples/sec to produce

$$x[n] = 4.2 \cos(0.75\pi n - 0.3)$$

This information suffices to determine $x(t)$, i.e., reconstruct the signal from its samples. First, we note that

$$2\pi \cdot \frac{f}{f_s} = 0.75\pi \quad \Rightarrow \quad f = (0.375)(400) = 150 \text{ Hz}$$

which is outside the given frequency range. From the previous example, the only alias of f in the range [600, 800] Hz is $f' = 650$ Hz. This is the correct frequency for $x(t)$, and

$$x(t) = 4.2 \cos(1300\pi t + 0.3)$$

Question: Why was the initial phase inverted (between $x(t)$ and $x[n]$)?

5. Analog-to-digital conversion involves sampling a signal $x(t)$ at a rate f_s and storing the samples in digital form (i.e., using finite precision). Digital-to-analog conversion is the reverse process of reconstructing $x(t)$ from its samples. If $x(t)$ is a sum of many sinusoidal components, then faithful reconstruction is impossible if aliasing has taken place, i.e., if two or more components of $x(t)$ have frequencies which are aliases of each other with respect of f_s .

Your task: Convince yourself that this is so by revisiting the previous example. If the same sample sequence $x[n]$ had represented the sum of two sinusoids in continuous time, at frequencies 150 and 650 Hz, could you have written an equation for $x(t)$ using the given formula for $x[n]$ only?

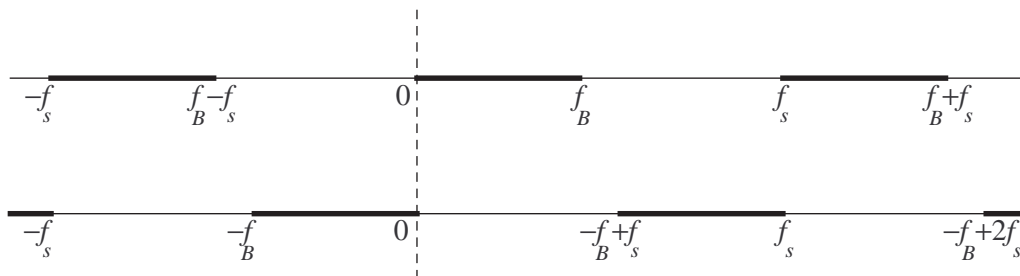
6. When sampling a signal $x(t)$ containing many different frequencies in the range $[0, f_B]$ Hz (B here stands for bandwidth), aliasing can be avoided if the sampling rate is greater than $2f_B$. One way of showing this is by plotting all aliases of frequencies in the given range $[0, f_B]$, using the equations derived earlier:

$$f' = f + kf_s \quad (\text{top axis})$$

and

$$f' = -f + kf_s \quad (\text{bottom axis})$$

(Plot on next page.) Each value of k corresponds to a translate of $[0, f_B]$ on the frequency axis. Aliasing is avoided when no two bands overlap (except at multiples of f_s). As can be seen from the figure, this is ensured if $f_s > 2f_B$.



Your task: Explain why aliasing occurs when $f_s = 2f_B - \delta$, where δ is positive amount less than, say, f_B . Find two frequencies in the interval $[0, f_B]$ that are aliases of each other with respect to f_s .

LECTURE 7

Topic: matrix-vector product; matrix of a linear transformation; matrix-matrix product

Textbook References: sections 2.1, 2.2.1, 2.2.2

Key Points:

- The matrix-vector product \mathbf{Ax} , where \mathbf{A} is a $m \times n$ matrix and \mathbf{x} is a n -dimensional column vector, is computed by taking the dot product of each row of \mathbf{A} with \mathbf{x} . The result is a m -dimensional column vector.
- For a fixed matrix \mathbf{A} , the product \mathbf{Ax} is linear in x :

$$\mathbf{A}(c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}) = c_1\mathbf{Ax}^{(1)} + c_2\mathbf{Ax}^{(2)}$$

In other words, \mathbf{A} acts as a linear transformation, or linear system, which maps n -dimensional vectors to m -dimensional ones.

- Every linear transformation, or linear system, $\mathbf{R}^n \rightarrow \mathbf{R}^m$ has a $m \times n$ matrix \mathbf{A} associated with it. Each column of \mathbf{A} is obtained by applying that transformation to the respective standard n -dimensional unit vector.
- If \mathbf{A} is $m \times p$ and \mathbf{B} is $p \times n$, then the product \mathbf{AB} is a $m \times n$ matrix whose $(i, j)^{\text{th}}$ element is the dot product of the i^{th} row of \mathbf{A} and the j^{th} column of \mathbf{B} .

Theory and Examples:

1. A $m \times n$ matrix consists of entries (or elements) a_{ij} , where i and j are the row and column indices, respectively. The space of all real-valued $m \times n$ matrices is denoted by $\mathbf{R}^{m \times n}$.
2. A column vector is a matrix consisting of one column only; a row vector is a matrix consisting of one row only. The transpose operator \cdot^T converts row vectors to column vectors and vice versa. By default, a lower-case boldface letter such as \mathbf{a} corresponds to a column vector. In situations where the orientation (row or column) of a vector is immaterial, we simply write

$$\mathbf{a} = (a_1, \dots, a_n)$$

which is a vector in \mathbf{R}^n .

3. The sum $\mathbf{S} = \mathbf{A} + \mathbf{B}$ of two matrices of the same dimension is obtained by adding respective entries together:

$$s_{ij} = a_{ij} + b_{ij}$$

The matrix $c\mathbf{A}$, where c is a real number, has the same dimensions as \mathbf{A} and is obtained by scaling each entry of \mathbf{A} by c .

4. If \mathbf{A} is a $m \times n$ matrix and \mathbf{x} is a $n \times 1$ (column) vector, then

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

is a $m \times 1$ vector such that

$$(\forall i) \quad y_i = \sum_{j=1}^n a_{ij}x_j$$

In other words, the i^{th} entry of \mathbf{y} is the dot product of the i^{th} row of \mathbf{A} with \mathbf{x} .

We will also view the product $\mathbf{y} = \mathbf{A}\mathbf{x}$ as a linear combination of the columns of \mathbf{A} with coefficients given by the (respective) entries of \mathbf{x} .

5. **Example.**

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & -1 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix} = \begin{bmatrix} 9 \\ -15 \end{bmatrix}$$

6. A vector of the form

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$$

where c_1 and c_2 are scalars, is known as a linear combination of the vectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

For a fixed matrix \mathbf{A} , the product $\mathbf{A}\mathbf{x}$ is linear in \mathbf{x} , i.e., it has the property that

$$\mathbf{A}(c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}) = c_1\mathbf{A}\mathbf{x}^{(1)} + c_2\mathbf{A}\mathbf{x}^{(2)}$$

for any vectors $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and scalars c_1 , c_2 . This is known as the superposition property, and is easily proved by considering the i^{th} entry on each side:

$$\sum_{j=1}^n a_{ij} (c_1x_j^{(1)} + c_2x_j^{(2)}) = c_1 \left(\sum_{j=1}^n a_{ij}x_j^{(1)} \right) + c_2 \left(\sum_{j=1}^n a_{ij}x_j^{(2)} \right)$$

We say that a $m \times n$ matrix \mathbf{A} represents a linear transformation of \mathbf{R}^n to \mathbf{R}^m . Such a linear transformation is also referred to as a linear system with n -dimensional input vector \mathbf{x} and m -dimensional output vector \mathbf{y} :



7. **Example.** Suppose the $2 \times n$ matrix \mathbf{A} and the n -dimensional column vectors \mathbf{u} and \mathbf{v} are such that

$$\mathbf{A}\mathbf{u} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{A}\mathbf{v} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Then

$$\mathbf{A}(\mathbf{u} - 2\mathbf{v}) = \begin{bmatrix} -11 \\ 8 \end{bmatrix}$$

8. Example. The linear transformation represented by the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

is such that

$$\mathbf{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ x_1 \\ x_2 \end{bmatrix}$$

Thus the effect of applying \mathbf{A} to an arbitrary vector \mathbf{x} is to shift the entries of \mathbf{x} up (or down) by two positions in a circular fashion. This linear transformation is an example of a permutation, and all permutations are linear.

9. Your task: You are given an image whose dimensions match those of a 36 inch (diagonal) display with an aspect ratio of 16 (horizontal) to 9 (vertical). You want to display the image on a 27 inch (diagonal) display with an aspect ratio of 4 (horizontal) to 3 (vertical) such that the image is as large as possible without distortion or cropping. Find the matrix

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = a\mathbf{I}$$

which accomplishes this. (Note: \mathbf{I} is the identity matrix.)

10. Conversely, every linear transformation $A : \mathbf{R}^n \rightarrow \mathbf{R}^m$ has a $m \times n$ matrix associated with it. This can be seen by expressing an arbitrary input vector \mathbf{x} as a linear combination of the standard unit vectors:

$$\mathbf{x} = x_1\mathbf{e}^{(1)} + \cdots + x_n\mathbf{e}^{(n)}$$

By linearity of $A(\cdot)$, the output vector $\mathbf{y} = A(\mathbf{x})$ equals

$$\mathbf{y} = x_1A(\mathbf{e}^{(1)}) + \cdots + x_nA(\mathbf{e}^{(n)})$$

If we form a $m \times n$ matrix $\mathbf{A} = [a_{ij}]$ using $A(\mathbf{e}^{(1)}), \dots, A(\mathbf{e}^{(n)})$ as its columns (in that order), then the output vector \mathbf{y} (above) is, in effect, a linear combination of the columns of \mathbf{A} with coefficients x_1, \dots, x_n . In other words,

$$(\forall i) \quad y_i = \sum_{j=1}^n a_{ij}x_j$$

and thus $\mathbf{y} = A(\mathbf{x})$ is also given by

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

11. Example. If the linear transformation $A(\cdot) : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is such that

$$A\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \quad A\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix} \quad \text{and} \quad A\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

then the matrix \mathbf{A} of $A(\cdot)$ is given by

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 0 & 4 & 1 \end{bmatrix}$$

- 12. Example.** Suppose now that $A(\cdot) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ represents the projection of a two-dimensional vector $\mathbf{x} = (x_1, x_2)$ onto the horizontal (i.e., x_1) axis. From vector geometry, we know that this is a linear transformation: the projection of a sum of (possibly scaled) vectors is the sum of their projections. We can therefore obtain the matrix \mathbf{A} by considering the result of applying $A(\cdot)$ to the two unit vectors $(1, 0)$ and $(0, 1)$. We have

$$\mathbf{A} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{A} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and consequently

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- 13. Example.** Similarly, the rotation of a two-dimensional vector through a fixed angle is linear: when two vectors are rotated through the same angle, their (possibly scaled) sum is also rotated through that angle. If \mathbf{B} is the matrix representing a counterclockwise rotation by 30° , then

$$\mathbf{B} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}, \quad \mathbf{B} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix},$$

and thus

$$\mathbf{B} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

Question: How were these values obtained?

- 14.** If \mathbf{A} is $m \times p$ and \mathbf{B} is $p \times n$, then the product \mathbf{AB} is the $m \times n$ matrix whose (i, j) th element is the dot product of the i th row of \mathbf{A} and the j th column of \mathbf{B} :

$$(AB)_{ij} = \sum_{k=1}^p a_{ik}b_{kj}$$

The number of columns of \mathbf{A} must be the same as the number of rows of \mathbf{B} (equal to p in this case, and also referred to as the inner dimension in the product).

- 15. Example.** If

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 1 & 5 \end{bmatrix}$$

then

$$\mathbf{AB} = \begin{bmatrix} 6 & 13 \\ 0 & -7 \end{bmatrix}$$

LECTURE 8

Topic: cascaded linear transformations; row and column selection; permutation matrices; matrix transpose

Textbook References: sections 2.2.3, 2.3

Key Points:

- \mathbf{AB} represents two linear systems connected in cascade (tandem, series), with system \mathbf{A} downstream from system \mathbf{B} .
- Associativity: $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \stackrel{\text{def}}{=} \mathbf{ABC}$
- Commutativity does not hold in general, i.e., $\mathbf{AB} \neq \mathbf{BA}$.
- To select the j^{th} column of a matrix, we right-multiply it by the j^{th} standard unit vector of the appropriate size. To select the i^{th} row of a matrix, we left-multiply it by the i^{th} standard unit vector.
- Transposition and multiplication: $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$
- A permutation matrix is a square matrix whose columns are distinct standard unit vectors. Right-multiplication by a permutation matrix results in column permutation; left-multiplication results in row permutation.
- If \mathbf{P} is a permutation matrix, then $\mathbf{P}^T \mathbf{P} = \mathbf{I}$ (the identity matrix). Thus the inverse of a permutation matrix is its transpose.

Theory and Examples:

1. Viewed as a linear transformation, the product \mathbf{AB} represents the cascade (series) connection of \mathbf{B} followed by \mathbf{A} , as shown below. This means that for a n -dimensional input vector \mathbf{x} , the output \mathbf{y} is given by

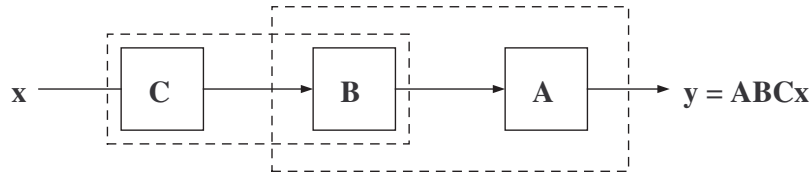
$$\mathbf{y} = (\mathbf{AB})\mathbf{x} = \mathbf{A}(\mathbf{Bx})$$



To see why this is so, take $\mathbf{x} = \mathbf{e}^{(j)}$, i.e., any standard unit vector in \mathbf{R}^n . Then

- $(\mathbf{AB})\mathbf{e}^{(j)}$ is the j^{th} column of the matrix \mathbf{AB} . By definition of \mathbf{AB} , the i^{th} entry in that column equals the dot product of the i^{th} row of \mathbf{A} and the j^{th} column of \mathbf{B} .
- $\mathbf{B}\mathbf{e}^{(j)}$ is the j^{th} column of \mathbf{B} . Thus the $(i, j)^{\text{th}}$ entry of $\mathbf{A}(\mathbf{B}\mathbf{e}^{(j)})$ equals the dot product of the i^{th} row of \mathbf{A} and the j^{th} column of \mathbf{B} .

Therefore $(\mathbf{AB})\mathbf{x} = \mathbf{A}(\mathbf{Bx})$ for $\mathbf{x} = \mathbf{e}^{(j)}$. This result extends to all vectors in \mathbf{R}^n . (**Question:** Why?)



2. Associativity: $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \stackrel{\text{def}}{=} \mathbf{ABC}$, with obvious extensions to products involving four or more matrices.
3. In general,

$$\mathbf{AB} \neq \mathbf{BA}$$

i.e., matrix multiplication is not commutative—even in cases where both products are well-defined and have the same dimensions (this happens if and only if both \mathbf{A} and \mathbf{B} are square matrices of the same dimensions). There are, however, exceptions.

4. **Your task:** Show that if \mathbf{A} represents projection onto the horizontal axis and \mathbf{B} represents counterclockwise rotation by an angle equal to $\pi/6$, then $\mathbf{AB} \neq \mathbf{BA}$. This can be shown both algebraically (i.e., by computing the products) and geometrically.
5. If \mathbf{A} is a $m \times n$ matrix and $\mathbf{e}^{(j)}$ is the j^{th} standard unit vector in $\mathbf{R}^{n \times 1}$, then

$$\mathbf{A}\mathbf{e}^{(j)} = j^{\text{th}} \text{ column of } \mathbf{A}$$

This follows directly from the definition of the matrix-vector product, where $\mathbf{A}\mathbf{x}$ is a linear combination of the columns of \mathbf{A} with coefficients given by the respective entries of \mathbf{x} .

6. **Example.** Suppose

$$\mathbf{A} = \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \kappa & \lambda & \mu & \nu \\ \phi & \chi & \psi & \omega \end{bmatrix}$$

Then

$$\mathbf{A} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \beta \\ \lambda \\ \chi \end{bmatrix}, \quad \mathbf{A} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \beta + 2\delta \\ \lambda + 2\nu \\ \chi + 2\omega \end{bmatrix} \quad \text{and} \quad \mathbf{A} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \beta & 2\delta \\ \lambda & 2\nu \\ \chi & 2\omega \end{bmatrix}$$

7. The columns of a $m \times n$ matrix \mathbf{A} are permuted by selecting all of them in a particular order. This amounts to taking the product \mathbf{AP} , where \mathbf{P} is a permutation matrix, i.e.,
- the columns of \mathbf{P} are the standard n -dimensional unit vectors (in any order); equivalently,
 - the rows of \mathbf{P} are the standard n -dimensional unit vectors (in any order).

8. Example.

$$\begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \kappa & \lambda & \mu & \nu \\ \phi & \chi & \psi & \omega \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \gamma & \delta & \alpha & \beta \\ \mu & \nu & \kappa & \lambda \\ \psi & \omega & \phi & \chi \end{bmatrix}$$

9. If \mathbf{A} is $m \times n$, its transpose \mathbf{A}^T is the $n \times m$ matrix whose $(i, j)^{\text{th}}$ entry equals the $(j, i)^{\text{th}}$ entry of \mathbf{A} . Thus the i^{th} row of \mathbf{A}^T equals the (transpose of) the i^{th} column of \mathbf{A} , and vice versa (for all i).

A square matrix \mathbf{A} is symmetric if $\mathbf{A} = \mathbf{A}^T$ (i.e., i^{th} row and i^{th} column are the same vector).

10. Example.

$$\begin{aligned} [a \ b \ c]^T &= \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ \begin{bmatrix} 5 & 1 \\ 2 & 4 \\ 6 & 7 \end{bmatrix}^T &= \begin{bmatrix} 5 & 2 & 6 \\ 1 & 4 & 7 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}^T &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \quad (\text{symmetric matrix}) \end{aligned}$$

11. The identity

$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$$

(assuming that the product \mathbf{AB} exists) can be shown by considering the $(i, j)^{\text{th}}$ element of $(\mathbf{AB})^T$, i.e., the $(j, i)^{\text{th}}$ element of \mathbf{AB} . This equals the dot product of the j^{th} row of \mathbf{A} and the i^{th} column of \mathbf{B} ; which is the same as the dot product of the j^{th} column of \mathbf{A}^T and the i^{th} row of \mathbf{B}^T . This is just the $(i, j)^{\text{th}}$ element of $\mathbf{B}^T \mathbf{A}^T$.

12. Row selections on a $m \times n$ matrix \mathbf{A} are accomplished by left-multiplying \mathbf{A} by the appropriate standard (row) unit vectors:

$$(\mathbf{e}^{(i)})^T \mathbf{A} = i^{\text{th}} \text{ row of } \mathbf{A}$$

If \mathbf{P} is a $m \times m$ permutation matrix, then \mathbf{PA} is a permutation of the rows of \mathbf{A} .

Question: If $n = m$ and \mathbf{AP} puts the columns of \mathbf{A} in a particular order, what should \mathbf{Q} be so that \mathbf{QA} results in the same reordering applied to row indices (instead of column indices)?

13. Example.

$$\begin{aligned} [0 \ 0 \ 1] \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \kappa & \lambda & \mu & \nu \\ \phi & \chi & \psi & \omega \end{bmatrix} &= [\phi \ \chi \ \psi \ \omega] \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \kappa & \lambda & \mu & \nu \\ \phi & \chi & \psi & \omega \end{bmatrix} &= \begin{bmatrix} \phi & \chi & \psi & \omega \\ \alpha & \beta & \gamma & \delta \\ \kappa & \lambda & \mu & \nu \end{bmatrix} \end{aligned}$$

14. If \mathbf{P} is a permutation matrix, then

$$\mathbf{P}^T \mathbf{P} = \mathbf{I}$$

This is shown by considering the $(i, j)^{\text{th}}$ element of $\mathbf{P}^T \mathbf{P}$, which equals the dot product of the i^{th} and j^{th} columns of \mathbf{P} . Since the columns of \mathbf{P} are distinct standard unit vectors, it follows that the dot product equals unity if $i = j$, zero otherwise.

LECTURE 9

Topic: introduction to matrix inversion; Gaussian elimination

Textbook References: sections 2.4, 2.5, 2.6

Key Points:

- A $n \times n$ matrix \mathbf{A} is nonsingular, i.e., it possesses an inverse \mathbf{A}^{-1} , if and only if the equation $\mathbf{Ax} = \mathbf{b}$ has a unique solution \mathbf{x} (equal to $\mathbf{A}^{-1}\mathbf{b}$) for every \mathbf{b} .
- $\mathbf{AA}^{-1} = \mathbf{I}$
- $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- A triangular matrix is nonsingular if all entries on its leading diagonal are nonzero. It can be inverted using forward or backward substitution.
- In Gaussian elimination, the $n \times n$ system of simultaneous equations $\mathbf{Ax} = \mathbf{b}$ is reduced to an upper triangular system $\mathbf{Ux} = \mathbf{c}$, which is then solved using backward substitution.
- The process of obtaining \mathbf{U} from \mathbf{A} (and \mathbf{c} from \mathbf{b}) is known as forward elimination, and proceeds in $n - 1$ steps. In the j^{th} step, the j^{th} equation is used for eliminating variable x_j from equations below it.
- A square matrix \mathbf{A} is nonsingular if and only if exactly one variable (no more) is eliminated in each step of the forward elimination process.

Theory and Examples:

1. Given the output of a linear system for which the matrix \mathbf{A} is known, is it possible to determine its input? This would involve solving the equation

$$\mathbf{Ax} = \mathbf{b}$$

where \mathbf{x} is the unknown input vector of dimension n , and $\mathbf{y} = \mathbf{b}$ is the observed output vector of dimension m .

2. If the entries of \mathbf{A} are random real numbers (with infinite precision), the following statements can be made about $\mathbf{Ax} = \mathbf{b}$ in probabilistic terms:
 - If $m < n$, any vector \mathbf{b} is almost certainly a valid system output, in which case a solution \mathbf{x} exists. The solution is not, however unique, and thus the “true” input vector cannot be determined.
 - If $m > n$, a randomly chosen vector \mathbf{b} is almost certainly not a valid system output, in which case no solution \mathbf{x} exists.
 - If $m = n$, every vector \mathbf{b} is almost certainly a valid system output corresponding to a unique system input \mathbf{x} . A solution \mathbf{x} therefore exists for every \mathbf{b} , and is unique.

3. There are, of course, exceptions to the “almost certain” rules given above. For example, in the case $m = n = 2$, taking \mathbf{A} as any of the matrices

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

will result in the equation $\mathbf{Ax} = \mathbf{b}$ having a solution for certain values of \mathbf{b} only, and that solution will not be unique.

(**Your task:** In each case, find a vector \mathbf{b} which is not expressible as a linear combination of the columns of \mathbf{A} . Also, show that $\mathbf{Ax} = \mathbf{0}$ has multiple solutions \mathbf{x} .)

4. The case $m = n$ is by far the most important, and is the key to understanding what happens in the other two cases ($m < n$ and $m > n$). A random choice of a $n \times n$ matrix \mathbf{A} is almost certain to have the property known as nonsingularity:

A square matrix \mathbf{A} is nonsingular if and only if the equation $\mathbf{Ax} = \mathbf{b}$ has a unique solution for every vector \mathbf{b} . It is singular otherwise.

5. If \mathbf{A} is nonsingular, every vector \mathbf{b} has a unique \mathbf{x} associated with it, such that $\mathbf{Ax} = \mathbf{b}$. It is not difficult to show that the mapping of \mathbf{b} 's to \mathbf{x} 's is linear, i.e., there is a matrix \mathbf{A}^{-1} such that

$$\mathbf{Ax} = \mathbf{b} \quad \Leftrightarrow \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Thus the inverse \mathbf{A}^{-1} exists if and only if \mathbf{A} is nonsingular.

6. Useful properties of the inverse of a nonsingular matrix \mathbf{A} :

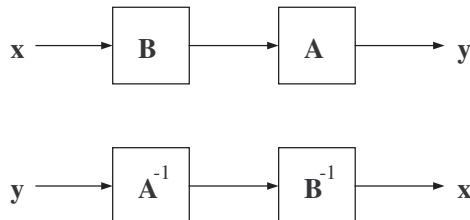
- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
- \mathbf{A}^T is also nonsingular, and

$$(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$$

This follows easily by taking the transpose of both sides of $\mathbf{AA}^{-1} = \mathbf{I}$.

- If \mathbf{B} is also nonsingular, then so is the product \mathbf{AB} , and

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$



7. Permutation matrices are nonsingular, and their inverse is given by their transpose: recall that $\mathbf{P}^T \mathbf{P} = \mathbf{I}$. Thus for example,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

8. A diagonal matrix \mathbf{D} is defined by the property that $d_{ij} = 0$ for $i \neq j$; i.e., all entries except those on the leading diagonal are zero. The equation $\mathbf{D}\mathbf{x} = \mathbf{b}$ is equivalent to

$$(\forall i) \quad d_{ii}x_i = b_i$$

and has a unique solution given by $x_i = d_{ii}^{-1}b_i$ for all i . This requires that all d_{ii} 's are nonzero, which is both a necessary and sufficient condition for a diagonal matrix \mathbf{D} to be nonsingular.

Thus for $\alpha\beta\gamma \neq 0$, we have

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}^{-1} = \begin{bmatrix} 1/\alpha & 0 & 0 \\ 0 & 1/\beta & 0 \\ 0 & 0 & 1/\gamma \end{bmatrix}$$

9. Triangular matrices are of particular importance in both the theoretical development of linear algebra and its practical, real-world applications. The defining feature of a lower (or upper) triangular matrix is that all elements above (or below, respectively) the leading diagonal are zero. If \mathbf{L} is lower triangular, the equation $\mathbf{L}\mathbf{x} = \mathbf{b}$ is expanded as

$$\begin{array}{rcl} \ell_{11}x_1 & = & b_1 \\ \ell_{21}x_1 + \ell_{22}x_2 & = & b_2 \\ \vdots & & \vdots \\ \ell_{n1}x_1 + \ell_{n2}x_2 + \cdots + \ell_{nn}x_n & = & b_n \end{array}$$

and is easily solved using the procedure known as forward substitution: x_1 is obtained from the first equation, then substituted into the second equation to obtain x_2 , and so on and so forth for the remaining x_i 's. Forward substitution works provided that all the leading diagonal coefficients ℓ_{ii} are nonzero. This condition is both necessary and sufficient for \mathbf{L} to be nonsingular.

Upper triangular matrices \mathbf{U} have similar properties. The equation $\mathbf{U}\mathbf{x} = \mathbf{b}$ is solved by backward substitution (i.e., $x_n \rightarrow x_{n-1} \rightarrow \dots \rightarrow x_1$) provided again that all the leading diagonal elements of \mathbf{U} are nonzero.

10. **Example.** Forward substitution in

$$\mathbf{L}\mathbf{x} = \begin{bmatrix} 2 & 0 & 0 \\ 5 & -1 & 0 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

gives

$$\begin{aligned}
2x_1 = b_1 &\Rightarrow x_1 = \frac{b_1}{2} \\
\frac{5b_1}{2} - x_2 = b_2 &\Rightarrow x_2 = \frac{5b_1}{2} - b_2 \\
\frac{b_1}{2} - 2\left(\frac{5b_1}{2} - b_2\right) + 3x_3 = b_3 &\Rightarrow x_3 = \frac{3b_1}{2} - \frac{2b_2}{3} + \frac{b_3}{3}
\end{aligned}$$

Therefore

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ 5/2 & -1 & 0 \\ 3/2 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \mathbf{L}^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 5/2 & -1 & 0 \\ 3/2 & -2/3 & 1/3 \end{bmatrix}$$

11. In general, the matrix-vector equation $\mathbf{Ax} = \mathbf{b}$, where \mathbf{A} is $n \times n$, represents a system of n linear equations in n unknowns x_1, \dots, x_n . If \mathbf{A} is nonsingular, the (unique) solution can be found using Gaussian elimination and its variants.
12. The principle behind Gaussian elimination is the equivalence of

$$\mathbf{Ax} = \mathbf{b} \quad \text{and} \quad \mathbf{MAx} = \mathbf{Mb}$$

where \mathbf{M} is a nonsingular $n \times n$ matrix. The solution is usually obtained in two phases, each consisting of $n - 1$ steps:

- the forward elimination phase reduces the system to an equivalent upper triangular system $\mathbf{Ux} = \mathbf{c}$;
 - the backward substitution phase solves the upper triangular system and produces \mathbf{x} .
13. We illustrate forward elimination for the 3×3 equation

$$\mathbf{Ax} = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 0 & -1 \\ -8 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ -10 \end{bmatrix}$$

Briefly:

- The first step eliminates the variable x_1 from the second and third equations, while the second step eliminates x_2 from the third equation.
- In the j^{th} step (where $j = 1, 2$), a multiple ($\times m_{ij}$) of the j^{th} row is added to each row below it (i.e., with row index $i > j$).
- The multipliers m_{ij} (listed in the column labeled m) are determined by taking ratios of entries in the j^{th} column. The denominator of each ratio is the $(j, j)^{\text{th}}$ entry, which is also known as the j^{th} pivot (shown in a box).

| m | x_1 | x_2 | x_3 | b |
|-----|---|--|-------|-----|
| | 2 | 1 | -1 | 6 |
| -2 | 4 | 0 | -1 | 6 |
| 4 | -8 | 2 | 3 | -10 |
| | 2 | 1 | -1 | 6 |
| | 0 | -2 | 1 | -6 |
| 3 | 0 | 6 | -1 | 14 |
| | 2 | 1 | -1 | 6 |
| | 0 | -2 | 1 | -6 |
| | 0 | 0 | 2 | -4 |

14. Forward elimination has thus resulted in the upper triangular system of equations

$$\begin{aligned}
 2x_1 + x_2 - x_3 &= 6 \\
 -2x_2 + x_3 &= -6 \\
 2x_3 &= -4
 \end{aligned}$$

which is easily solved by backward substitution to produce $x_3 = -2 \rightarrow x_2 = 2 \rightarrow x_1 = 1$, i.e., $\mathbf{x} = [1 \quad 2 \quad -2]^T$.

15. Gaussian elimination can be completed (as outlined above) provided all diagonal elements, or pivots, obtained in the forward elimination phase are nonzero. If a zero pivot is encountered in the $j^{\text{th}} \leq n - 1$ step, the j^{th} equation cannot be used to eliminate x_j from the equations below it. This difficulty is resolved by a simple row swap provided one of those equations has a nonzero coefficient for x_j .

- If all zero pivots can be resolved by row swaps, and the the final form of the n^{th} equation has a nonzero coefficient for x_n , then $\mathbf{Ax} = \mathbf{b}$ has a unique solution \mathbf{x} for every \mathbf{b} , i.e., the matrix \mathbf{A} is nonsingular.
- Otherwise—if at some j^{th} step all remaining (unused) equations have a nonzero coefficient for x_j , or if the final form of the n^{th} equation has a zero coefficient for x_n —the matrix \mathbf{A} is singular.

LECTURE 10

Topics: inner products, norms and angles; projection

Textbook References: sections 2.10.1–2.10.4, 2.12.1

Key Points:

- The norm, or length, of a real-valued vector \mathbf{a} is given by

$$\|\mathbf{a}\| = \langle \mathbf{a}, \mathbf{a} \rangle^{1/2},$$

where $\langle \cdot, \cdot \rangle$ denotes inner (dot) product.

- The angle θ between two vectors \mathbf{a} and \mathbf{b} satisfies

$$\cos \theta = \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}$$

- The (orthogonal) projection of \mathbf{b} on \mathbf{a} is the vector $\lambda \mathbf{a}$ such that $\mathbf{b} - \lambda \mathbf{a}$ is orthogonal to \mathbf{a} , i.e., $\langle \mathbf{b} - \lambda \mathbf{a}, \mathbf{a} \rangle = 0$. Therefore

$$\lambda = \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{\|\mathbf{a}\|^2}$$

- The range (or column space) $\mathcal{R}(\mathbf{V})$ of the $m \times n$ matrix

$$\mathbf{V} = [\mathbf{v}^{(1)} \quad \dots \quad \mathbf{v}^{(n)}]$$

consists of all linear combinations of columns of \mathbf{V} , i.e., vectors of the form $\mathbf{V}\mathbf{c}$.

- The projection of a m -dimensional vector \mathbf{s} onto the range $\mathcal{R}(\mathbf{V})$ of a matrix \mathbf{V} is the unique vector $\hat{\mathbf{s}}$ in $\mathcal{R}(\mathbf{V})$ with the property that $\mathbf{s} - \hat{\mathbf{s}}$ is orthogonal to every column of \mathbf{V} . It is the point in $\mathcal{R}(\mathbf{V})$ closest to \mathbf{s} .
- If the columns of \mathbf{V} are pairwise orthogonal, then projection onto $\mathcal{R}(\mathbf{V})$ amounts simply to projecting onto each column, then summing the projections.
- A $m \times m$ matrix \mathbf{V} consisting of nontrivial ($\neq \mathbf{0}$) pairwise orthogonal vectors is nonsingular. Any vector \mathbf{s} in \mathbf{R}^m can be expressed as $\mathbf{s} = \mathbf{V}\mathbf{c}$, where \mathbf{c} is found by projecting \mathbf{s} onto each of the columns of \mathbf{V} .

Theory and Examples:

1. The $m \times m$ system $\mathbf{A}\mathbf{x} = \mathbf{b}$ can be solved in a particularly simple way (without Gaussian elimination) if any two *columns* of \mathbf{A} have zero dot product. To determine the value of any x_j , we take dot products of both sides of the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ with the j^{th} column $\mathbf{a} = \mathbf{a}^{(j)}$ (here $\mathbf{0}_i$ denotes a column vector of i zeros):

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{b} \\ (\mathbf{a}^T \mathbf{A})\mathbf{x} &= \mathbf{a}^T \mathbf{b} \\ [\mathbf{0}_{j-1}^T \quad \mathbf{a}^T \mathbf{a} \quad \mathbf{0}_{m-j}^T] \mathbf{x} &= \mathbf{a}^T \mathbf{b} \end{aligned}$$

In other words,

$$x_j = \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}}$$

provided $\mathbf{a}^T \mathbf{a} \neq 0$, i.e., \mathbf{a} is not an all-zeros vector.

2. For real-valued vectors, the dot product is also known as inner product and plays an important role in the development of vector geometry. If \mathbf{a} and \mathbf{b} are real and m -dimensional, their inner product is denoted and given by

$$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^T \mathbf{b} = \sum_{i=1}^m a_i b_i$$

It is commutative, i.e.,

$$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^T \mathbf{b} = \mathbf{b}^T \mathbf{a} = \langle \mathbf{b}, \mathbf{a} \rangle,$$

and also linear in one of the two arguments when the other one is kept constant:

$$\langle c_1 \mathbf{a}^{(1)} + c_2 \mathbf{a}^{(2)}, \mathbf{b} \rangle = c_1 \langle \mathbf{a}^{(1)}, \mathbf{b} \rangle + c_2 \langle \mathbf{a}^{(2)}, \mathbf{b} \rangle$$

3. The norm, or length, of \mathbf{a} is defined by

$$\|\mathbf{a}\| = \langle \mathbf{a}, \mathbf{a} \rangle^{1/2} = \left(\sum_{i=1}^m a_i^2 \right)^{1/2}$$

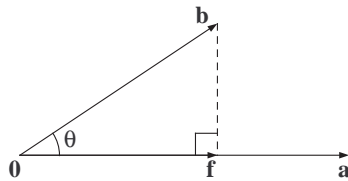
and is strictly positive unless $\mathbf{a} = \mathbf{0}$. Scaling \mathbf{a} by a constant c results in $\|\mathbf{a}\|$ being scaled by $|c|$.

4. Two vectors \mathbf{a} and \mathbf{b} are orthogonal (denoted by $\mathbf{a} \perp \mathbf{b}$) if $\langle \mathbf{a}, \mathbf{b} \rangle = 0$. This notion of orthogonality agrees with geometry, since it implies that

$$\begin{aligned} \|\mathbf{a} + \mathbf{b}\|^2 &= \langle \mathbf{a} + \mathbf{b}, \mathbf{a} + \mathbf{b} \rangle \\ &= \langle \mathbf{a}, \mathbf{a} \rangle + 2\langle \mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{b}, \mathbf{b} \rangle = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \end{aligned}$$

5. From geometry, we know that the (orthogonal) projection of \mathbf{b} onto \mathbf{a} is the vector $\mathbf{f} = \lambda \mathbf{a}$ such that $\mathbf{b} - \mathbf{f}$ is orthogonal to \mathbf{a} . This condition determines the value of λ :

$$\langle \mathbf{b} - \lambda \mathbf{a}, \mathbf{a} \rangle = 0 \quad \Leftrightarrow \quad \lambda = \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{\|\mathbf{a}\|^2}$$



6. We also have $\|\mathbf{b}\| \cos \theta = \lambda \|\mathbf{a}\|$, and therefore

$$\cos \theta = \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}$$

(assuming that neither \mathbf{a} nor \mathbf{b} is an all-zeros vector). The range of values of θ is $[0, \pi]$.

7. **Example.** Consider the three vectors

$$\begin{aligned} \mathbf{v}^{(1)} &= [5 \quad 1 \quad 3 \quad 1]^T \\ \mathbf{v}^{(2)} &= [1 \quad 3 \quad -1 \quad -5]^T \\ \mathbf{v}^{(3)} &= [-6 \quad 6 \quad -6 \quad 6]^T \end{aligned}$$

Then:

- $\|\mathbf{v}^{(1)}\| = \|\mathbf{v}^{(2)}\| = \|\mathbf{v}^{(3)}\|/2 = 6$
 - $\mathbf{v}^{(1)} \perp \mathbf{v}^{(2)}$
 - the projection of $\mathbf{v}^{(3)}$ onto $\mathbf{v}^{(1)}$ equals $-\mathbf{v}^{(1)}$
 - the angle between $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(3)}$ equals $2\pi/3$
8. As we saw earlier, the $m \times m$ equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ can be solved easily using inner products if the columns of \mathbf{A} are pairwise orthogonal (and nontrivial, i.e., none is the all-zeros vector $\mathbf{0}$). We will rewrite this equation as

$$\mathbf{V}\mathbf{c} = \mathbf{s},$$

where, instead of considering \mathbf{V} as a linear system (with input \mathbf{c} and output \mathbf{s}), we will adopt a different view. Specifically,

$$\mathbf{V} = [\mathbf{v}^{(1)} \quad \dots \quad \mathbf{v}^{(n)}]$$

is an ordered set of m *orthogonal reference* vectors $\mathbf{v}^{(j)}$, which when properly combined linearly—with coefficients c_j —can produce any m -dimensional vector \mathbf{s} . The equation $\mathbf{V}\mathbf{c} = \mathbf{s}$ (with \mathbf{c} unknown) is therefore a *signal representation* problem.

9. If the matrix \mathbf{V} has $n < m$ orthogonal columns, then the set of linear combinations of the form $\mathbf{V}\mathbf{c}$ —known as the range of \mathbf{V} and denoted by $\mathcal{R}(\mathbf{V})$ —does not include every vector in \mathbf{R}^m . In geometric terms, $\mathcal{R}(\mathbf{V})$ is a “flat” subset of \mathbf{R}^m containing the origin, also known as a linear subspace. For example, if \mathbf{V} consists of $n = 2$ nontrivial orthogonal columns of length $m = 3$, then $\mathcal{R}(\mathbf{V})$ is a two-dimensional plane in the three-dimensional Euclidean space \mathbf{R}^3 .

In such cases, a generic vector $\mathbf{s} \in \mathbf{R}^m$ cannot be represented exactly as $\mathbf{V}\mathbf{c}$; at best, $\hat{\mathbf{s}} = \mathbf{V}\mathbf{c}$ approximates \mathbf{s} . The *linear least squares approximation* of \mathbf{s} by the columns of \mathbf{V} is defined by the property that the distance between \mathbf{s} and $\hat{\mathbf{s}}$ is a minimum. It is therefore obtained by minimizing

$$\|\mathbf{s} - \mathbf{V}\mathbf{c}\|$$

over all n -dimensional vectors \mathbf{c} .

10. The minimization can be performed equivalently on the squared distance

$$\begin{aligned}\|\mathbf{s} - \mathbf{V}\mathbf{c}\|^2 &= \langle \mathbf{s} - \mathbf{V}\mathbf{c}, \mathbf{s} - \mathbf{V}\mathbf{c} \rangle \\ &= \|\mathbf{s}\|^2 - 2\mathbf{c}^T \mathbf{V}^T \mathbf{s} + \mathbf{c}^T \mathbf{V}^T \mathbf{V} \mathbf{c} \\ &= \|\mathbf{s}\|^2 + \sum_{j=1}^n \left(\|\mathbf{v}^{(j)}\|^2 c_j^2 - 2\langle \mathbf{v}^{(j)}, \mathbf{s} \rangle c_j \right)\end{aligned}$$

where the last equality was obtained using the fact that $\mathbf{V}^T \mathbf{V}$ is a diagonal matrix with each (j, j) th entry given by $\|\mathbf{v}^{(j)}\|^2$. It suffices to minimize each term in the \sum_j sum, which is a quadratic in c_j . Noting that

$$t^2 - at = t(t - a)$$

is minimized at $t = a/2$, we obtain the solution

$$c_j = \frac{\langle \mathbf{v}^{(j)}, \mathbf{s} \rangle}{\|\mathbf{v}^{(j)}\|^2} \quad (1 \leq j \leq n)$$

11. The coefficients c_1, \dots, c_j are the same as those used for projecting \mathbf{s} onto $\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n)}$, respectively. Thus

$$\hat{\mathbf{s}} = \mathbf{V}\mathbf{c} = \sum_{j=1}^n c_j \mathbf{v}^{(j)}$$

is the sum of the projections of \mathbf{s} onto each one of the columns of \mathbf{V} .

The difference $\mathbf{s} - \hat{\mathbf{s}}$, known as the error vector, is orthogonal to each of the columns of \mathbf{V} . This follows from the fact that for a given j , $\mathbf{s} - c_j \mathbf{v}^{(j)}$ is orthogonal to $\mathbf{v}^{(j)}$; and $\sum_{i \neq j} c_i \mathbf{v}^{(i)}$ is orthogonal to $\mathbf{v}^{(j)}$ also.

These results are extensions of familiar facts in three-dimensional Euclidean geometry:

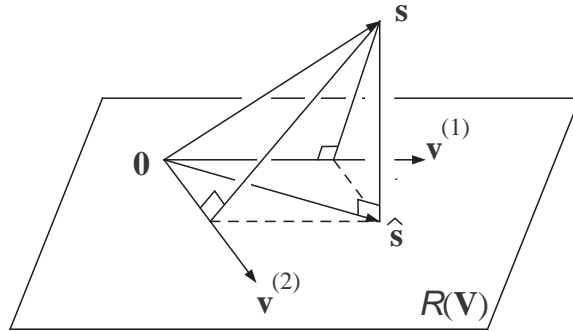
- The distance of a fixed point \mathbf{s} from variable point on a (two-dimensional) plane containing the origin is minimized at the projection of \mathbf{s} onto that plane; the projection the unique vector $\hat{\mathbf{s}}$ such that $\mathbf{s} - \hat{\mathbf{s}}$ is orthogonal to every vector on the plane.
- If $\mathbf{v}^{(1)} \neq \mathbf{0}$ and $\mathbf{v}^{(2)} \neq \mathbf{0}$ are two orthogonal vectors on the plane, then the projection $\hat{\mathbf{s}}$ of \mathbf{s} onto the plane can be formed by taking the vector sum of the projections of $\hat{\mathbf{s}}$ onto $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$.

12. **Example.** The columns of

$$\mathbf{V} = [\mathbf{v}^{(1)} \quad \mathbf{v}^{(2)}] = \begin{bmatrix} 1 & 3 \\ -2 & 4 \\ 1 & 5 \end{bmatrix}$$

are orthogonal and such that $\|\mathbf{v}^{(1)}\|^2 = 6$ and $\|\mathbf{v}^{(2)}\|^2 = 50$. The projection of $\mathbf{s} = [-6 \ 2 \ 7]^T$ onto $\mathcal{R}(\mathbf{V})$ is given by

$$\hat{\mathbf{s}} = c_1 \mathbf{v}^{(1)} + c_2 \mathbf{v}^{(2)},$$



where

$$c_1 = \frac{-6 - 4 + 7}{6} = -\frac{1}{2} \quad \text{and} \quad c_2 = \frac{-18 + 8 + 35}{50} = \frac{1}{2}$$

Thus $\hat{\mathbf{s}} = [1 \ 3 \ 2]^T$. The error vector is given by $\mathbf{s} - \hat{\mathbf{s}} = [-7 \ -1 \ 5]^T$. It is orthogonal to both $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$.

LECTURE 11

Topics: complex matrices and vectors

Textbook References: 2.13

Key Points:

- For complex vectors, the inner product and norm are defined by

$$\begin{aligned}\langle \mathbf{v}, \mathbf{w} \rangle &= \sum_{i=1}^m v_i^* w_i = (\mathbf{v}^*)^T \mathbf{w} \\ \|\mathbf{v}\| &= \langle \mathbf{v}, \mathbf{v} \rangle^{1/2} = \left(\sum_{i=1}^m |v_i|^2 \right)^{1/2}\end{aligned}$$

$(\mathbf{v}^*)^T$ is also denoted as \mathbf{v}^H .

- The relationship between projection, orthogonality and linear least squares approximation is the same for complex vectors as for real vectors.

Theory and Examples:

1. Switching from real-valued matrices and vectors to complex-valued ones is straightforward.
 - There are no changes in basic algebraic operations (addition, scaling and multiplication) other than the obvious use of complex algebra.
 - Linear independence is defined as before, allowing complex coefficients in linear combinations.
 - Gaussian elimination follows the same procedure as before, but involves more computations.
2. The norm of a complex-valued vector $\mathbf{v} \in \mathbf{C}^m$ is defined by

$$\|\mathbf{v}\| = \left(\sum_{i=1}^m |v_i|^2 \right)^{1/2}$$

Thus $\|\mathbf{v}\|^2$ is the sum of squares of all real and imaginary parts contained in \mathbf{v} . Since $|z|^2 = z^* z$, we also have

$$\|\mathbf{v}\|^2 = \sum_{i=1}^m v_i^* v_i = (\mathbf{v}^*)^T \mathbf{v}$$

The combination of the complex conjugate (\cdot^*) and transpose (\cdot^T) operators (in either order) is the *conjugate transpose*, or *Hermitian*, operator \cdot^H . Thus

$$\|\mathbf{v}\|^2 = \mathbf{v}^H \mathbf{v}$$

3. Next we define the inner product $\langle \mathbf{v}, \mathbf{w} \rangle$. In order to have $\langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{v}\|^2$, we set

$$\langle \mathbf{v}, \mathbf{w} \rangle \stackrel{\text{def}}{=} \sum_{i=1}^m v_i^* w_i = \mathbf{v}^H \mathbf{w}$$

(**Question:** What would be an alternative definition?)

Care must be exercised in computing inner products, particularly when complex valued coefficients are involved. Note, in particular, that $\langle \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle^*$ and

$$\langle \alpha \mathbf{v}, \beta \mathbf{w} \rangle = (\alpha^* \beta) \langle \mathbf{v}, \mathbf{w} \rangle \quad (\Rightarrow \quad \|\alpha \mathbf{v}\|^2 = |\alpha|^2 \cdot \|\mathbf{v}\|^2)$$

4. Orthogonality of two complex-valued vectors is defined as in the real-valued case, i.e., by the property that their inner product equals zero. Projections are developed in the same manner:

- The projection of \mathbf{s} onto $\mathbf{v} \neq \mathbf{0}$ is given by $c\mathbf{v}$, where

$$\langle \mathbf{v}, \mathbf{s} - c\mathbf{v} \rangle = 0 \quad \Leftrightarrow \quad c = \frac{\langle \mathbf{v}, \mathbf{s} \rangle}{\|\mathbf{v}\|^2}$$

- If the $m \times n$ matrix \mathbf{V} has nontrivial columns $\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n)}$ which are pairwise orthogonal, then the projection of \mathbf{s} onto the range of \mathbf{V} is given by $\hat{\mathbf{s}} = \mathbf{V}\mathbf{c}$, where

$$c_j = \frac{\langle \mathbf{v}^{(j)}, \mathbf{s} \rangle}{\|\mathbf{v}^{(j)}\|^2} \quad (1 \leq j \leq n)$$

- The projection of \mathbf{s} onto the range of \mathbf{V} is also the linear least-squares approximation of \mathbf{s} by the columns of \mathbf{V} , i.e., it minimizes the squared error norm

$$\|\mathbf{s} - \mathbf{V}\mathbf{c}\|^2$$

- If $n = m$, then $\mathbf{s} = \hat{\mathbf{s}} = \mathbf{V}\mathbf{c}$, i.e., projection yields an exact representation.

5. **Example.** (**Your task:** Verify the results.) The matrix

$$\mathbf{V} = [\mathbf{v}^{(0)} \quad \mathbf{v}^{(1)} \quad \mathbf{v}^{(2)} \quad \mathbf{v}^{(3)}] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

has orthogonal columns, each column having norm equal to 2:

$$\mathbf{V}^H \mathbf{V} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 4\mathbf{I}$$

The inverse of \mathbf{V} is given by

$$\mathbf{V}^{-1} = \frac{1}{4} \mathbf{V}^H \quad (\text{Question: Why?})$$

6. (Example, continued.) Any vector $\mathbf{s} \in \mathbf{C}^4$ can be expressed as

$$\mathbf{s} = \mathbf{V}\mathbf{c}$$

where the coefficients in \mathbf{c} are found by projecting \mathbf{s} onto the appropriate columns in \mathbf{V} . In this case, all columns have the same norm (equal to 2), so we simply have

$$\mathbf{c} = \mathbf{V}^H \mathbf{s} / 4$$

For instance,

$$\mathbf{s} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 9 \end{bmatrix} \Rightarrow \mathbf{c} = \begin{bmatrix} 2 \\ -1 + 2j \\ -3 \\ -1 - 2j \end{bmatrix}$$

(**Your task:** Verify the algebra.) Therefore

$$\mathbf{s} = 2\mathbf{v}^{(0)} + (-1 + 2j)\mathbf{v}^{(1)} - 3\mathbf{v}^{(2)} + (-1 - 2j)\mathbf{v}^{(3)}$$

7. Expressing a vector \mathbf{s} in the form $\mathbf{V}\mathbf{c}$, where \mathbf{V} is a square matrix with orthogonal columns, can be viewed as a change of coordinates. The new coordinates \mathbf{c} are in terms of the columns of \mathbf{V} , i.e., the new coordinate axes. Since these axes are orthogonal, we can again compute the length of \mathbf{s} using Pythagoras' theorem, provided we take the length of each $\mathbf{v}^{(j)}$ into account. More precisely,

$$\|\mathbf{s}\|^2 = \mathbf{s}^H \mathbf{s} = \mathbf{c}^H \mathbf{V}^H \mathbf{V} \mathbf{c} = \sum_{j=1}^m c_j^* c_j \cdot \|\mathbf{v}^{(j)}\|^2 = \sum_{j=1}^m |c_j|^2 \cdot \|\mathbf{v}^{(j)}\|^2$$

where the third equality is due to the fact that $\mathbf{V}^H \mathbf{V}$ is diagonal with each (j, j) th entry equal to $\|\mathbf{v}^{(j)}\|^2$.

The same concept applies to the inner products of two vectors, i.e., it can be computed using the new coordinates. Thus if

$$\mathbf{s} = \mathbf{V}\mathbf{c} \quad \text{and} \quad \mathbf{x} = \mathbf{V}\mathbf{d},$$

then

$$\langle \mathbf{s}, \mathbf{x} \rangle = \mathbf{s}^H \mathbf{x} = \mathbf{c}^H \mathbf{V}^H \mathbf{V} \mathbf{d} = \sum_{j=1}^m c_j^* d_j \cdot \|\mathbf{v}^{(j)}\|^2$$

8. In the previous example,

$$\|\mathbf{s}\|^2 = (-3)^2 + 1^2 + 1^2 + 9^2 = 4[2^2 + ((-1)^2 + 2^2) + (-3)^2 + ((-1)^2 + (-2)^2)] = 92$$

Also, if

$$\mathbf{x} = \mathbf{v}^{(0)} - \mathbf{v}^{(1)} + 3\mathbf{v}^{(2)} + \mathbf{v}^{(3)},$$

then

$$\langle \mathbf{s}, \mathbf{x} \rangle = 4[(2)(1) + (-1 - 2j)(-1) + (-3)(3) + (-1 + 2j)(1)] = -28 + 16j$$

(**Your task:** Compute this inner product directly using the entries of \mathbf{s} and \mathbf{x} .)

9. The matrix \mathbf{V} of the previous example plays a special role in signal analysis.

- Expressing an arbitrary vector as a linear combination of the orthogonal columns of \mathbf{V} involves the computation of four inner products; those inner products are referred to as a discrete Fourier transform (DFT) of length $N = 4$.
- The columns of \mathbf{V} , namely

$$\mathbf{v}^{(0)}, \quad \mathbf{v}^{(1)}, \quad \mathbf{v}^{(2)} \quad \text{and} \quad \mathbf{v}^{(3)}$$

are formed by raising, respectively,

$$j^0 = 1, \quad j, \quad j^2 = -1, \quad \text{and} \quad j^3 = -j$$

to powers $n = 0, 1, 2, 3$. This means that the n^{th} entry of each column is given by $e^{j\omega n}$, where

$$\omega = 0, \quad \pi/2, \quad \pi \quad \text{and} \quad 3\pi/2 \quad (\text{respectively})$$

- The four columns of \mathbf{V} are therefore complex sinusoidal vectors whose frequencies are uniformly spaced over the interval $[0, 2\pi)$ —or around the unit circle, depending on how you look at them.
- The four sinusoids involved here have fundamental periods 1, 4, 2 and 4. Therefore they all satisfy $x[n] = x[n+4]$. There are no other values of $\omega \in [0, 2\pi)$ for which $e^{j\omega n}$ repeats itself every four samples.

10. As it turns out, the matrix \mathbf{V} can be generalized to N dimensions, where it is defined as the $N \times N$ matrix whose k^{th} column $\mathbf{v}^{(k)}$ is a complex sinusoidal vector of frequency $k(2\pi/N)$ and initial (i.e., topmost) value equal to 1. These mutually orthogonal vectors serve as building blocks in discrete Fourier analysis, since any N -dimensional signal vector \mathbf{s} can be expressed as their linear combination with coefficients determined by the inner products $\langle \mathbf{v}^{(k)}, \mathbf{s} \rangle$. These N inner products form the discrete Fourier transform of \mathbf{s} .

LECTURE 12

Topic: introduction to the discrete Fourier transform; basic examples

Textbook References: sections 3.1, 3.2

Key Points:

- The columns of the $N \times N$ matrix \mathbf{V} , defined by $V_{nk} = e^{j(2\pi/N)kn}$, are sinusoidal vectors with frequencies which are (all the distinct) multiples of $2\pi/N$. These vectors are orthogonal and such that

$$\mathbf{V}^H \mathbf{V} = N\mathbf{I}$$

- Any signal vector \mathbf{s} can be expressed as a linear combination of such sinusoids: $\mathbf{s} = \mathbf{V}\mathbf{c}$, where $\mathbf{c} = \mathbf{V}^H \mathbf{s}/N$.

| Time Domain | | Frequency Domain |
|---|--------------------------------|--|
| \mathbf{s} | $\xleftrightarrow{\text{DFT}}$ | \mathbf{S} |
| $\mathbf{s} = \frac{1}{N} \mathbf{V}\mathbf{S}$ | \longleftrightarrow | $\mathbf{S} = \mathbf{V}^H \mathbf{s}$ |
| (Synthesis Equation) | | (Analysis Equation) |
| $\mathbf{s} = \text{ifft}(\mathbf{S})$ | | $\mathbf{S} = \text{fft}(\mathbf{s})$ |

- The N -point vector \mathbf{s} is a complex sinusoid of frequency $\omega = 2k\pi/N$ if and only its DFT (or spectrum) \mathbf{S} contains all zeros except for $S[k] \neq 0$.

Theory and Examples:

1. The discrete Fourier transform (DFT) is a powerful computational tool. It allows us to resolve finite-dimensional signal vectors into sinusoids of different frequencies, some of which may be more prominent than others. For example, the 200-point signal vector \mathbf{s} constructed in

```
n = (0:199).';
s = 4.7*cos(0.12*pi*n-1.3) + ...
    3.8*cos(0.19*pi*n+0.8) + ...
    5.1*cos(0.23*pi*n+2.4) + ...
    2.0*randn(size(n));
plot(n,s)
```

is the sum of three real-valued sinusoids plus noise. This is not at all that obvious from a plot of the signal. If, however, we compute

```
S = fft(s);
bar(abs(S))
```

then we obtain a symmetric graph with three clear peaks on either half. These peaks correspond to the main frequency components at $\omega = (0.12)\pi$, $(0.19)\pi$ and $(0.23)\pi$. Based on this information, we can characterize the signal \mathbf{s} as a sum of three sinusoids plus noise.

2. The MATLAB command `FFT` computes the DFT \mathbf{S} of a signal vector \mathbf{s} . \mathbf{S} is a complex-valued vector of the same size as \mathbf{s} , whose entries reflect the relative amounts of the sinusoidal components of \mathbf{s} . The number of frequencies involved is the same as the size of \mathbf{s} , and as we will soon see, these frequencies are uniformly spaced in $[0, 2\pi)$.
3. The principle behind the DFT is already familiar to us: a vector \mathbf{s} of size N is expressed as a linear combination of the columns of a $N \times N$ matrix \mathbf{V} , i.e.,

$$\mathbf{s} = \mathbf{V}\mathbf{c}$$

Each column of \mathbf{V} is a complex-valued sinusoid, i.e., it is given by the formula $e^{j\omega n}$, where n is the row index representing time. No two columns use the same frequency ω ; moreover, the N frequencies are chosen such that the columns of \mathbf{V} are orthogonal:

$$\mathbf{V}^H\mathbf{V} = \text{diagonal matrix}$$

This simplifies the computation of the coefficients \mathbf{c} , since

$$(\mathbf{V}^H\mathbf{V})\mathbf{c} = \mathbf{V}^H\mathbf{s}$$

reduces to a diagonal system of equations. (The DFT \mathbf{S} is defined simply as $N\mathbf{c}$.)

4. We have already encountered the matrix \mathbf{V} in the case $N = 4$:

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}^{(0)} & \mathbf{v}^{(1)} & \mathbf{v}^{(2)} & \mathbf{v}^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

Each column (from left to right) is a complex-valued sinusoid of frequency

$$\omega = 0, \quad \pi/2, \quad \pi \quad \text{and} \quad 3\pi/2$$

Note that the four frequencies are uniformly spaced over $[0, 2\pi)$ (or the unit circle).

5. In the general case (i.e., for arbitrary N), the frequencies of the N sinusoids are also uniformly spaced over $[0, 2\pi)$. Specifically, the k^{th} column of \mathbf{V} (starting with $k = 0$) is a complex-valued sinusoid of frequency

$$\omega = k \cdot \frac{2\pi}{N}$$

This choice of ω makes the sinusoid $e^{j\omega n}$ periodic with period equal to N or a submultiple thereof. No other values of ω result in this property.

If we set $v = e^{j(2\pi/N)}$, then the columns of \mathbf{V} are formed by raising the complex constants

$$v^0 = 1, \quad v, \quad v^2, \quad \dots, \quad v^{N-1}$$

to powers $n = 0 : N - 1$. These N complex numbers are precisely the roots of $z^N = 1$.

6. The orthogonality of the columns of \mathbf{V} can be shown using the geometric sum formula

$$1 + z + z^2 + \cdots + z^{N-1} = \begin{cases} (1 - z^N)/(1 - z), & z \neq 1; \\ N, & z = 1. \end{cases}$$

If $\mathbf{v}^{(k)}$ and $\mathbf{v}^{(\ell)}$ are two different columns of \mathbf{V} , then

$$\langle \mathbf{v}^{(k)}, \mathbf{v}^{(\ell)} \rangle = \sum_{n=0}^{N-1} v^{(\ell-k)n} = 0$$

(since $v^{(\ell-k)N} = 1$). On the other hand,

$$\langle \mathbf{v}^{(k)}, \mathbf{v}^{(k)} \rangle = \sum_{n=0}^{N-1} v^0 = N$$

This establishes the general result

$$\mathbf{V}^H \mathbf{V} = N\mathbf{I}$$

7. **Example.** In the case $N = 6$, the Fourier frequencies are

$$\omega = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$$

and the columns of \mathbf{V} are sinusoids at these frequencies:

$$\mathbf{V} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} + j\frac{\sqrt{3}}{2} & -\frac{1}{2} + j\frac{\sqrt{3}}{2} & -1 & -\frac{1}{2} - j\frac{\sqrt{3}}{2} & \frac{1}{2} - j\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} + j\frac{\sqrt{3}}{2} & -\frac{1}{2} - j\frac{\sqrt{3}}{2} & 1 & -\frac{1}{2} + j\frac{\sqrt{3}}{2} & -\frac{1}{2} - j\frac{\sqrt{3}}{2} \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1}{2} - j\frac{\sqrt{3}}{2} & -\frac{1}{2} + j\frac{\sqrt{3}}{2} & 1 & -\frac{1}{2} - j\frac{\sqrt{3}}{2} & -\frac{1}{2} + j\frac{\sqrt{3}}{2} \\ 1 & \frac{1}{2} - j\frac{\sqrt{3}}{2} & -\frac{1}{2} - j\frac{\sqrt{3}}{2} & -1 & -\frac{1}{2} + j\frac{\sqrt{3}}{2} & \frac{1}{2} + j\frac{\sqrt{3}}{2} \end{bmatrix}$$

In this case, $v = e^{j(2\pi/6)} = e^{j\pi/3}$ and $V_{nk} = v^{nk} = e^{j(\pi/3)kn}$.

If $\mathbf{s} = [1 \ 2 \ 3 \ 6 \ 5 \ 4]^T$, then the coefficients \mathbf{c} in $\mathbf{s} = \mathbf{V}\mathbf{c}$ are quickly found using

$$\mathbf{c} = \text{fft}([1 \ 2 \ 3 \ 6 \ 5 \ 4] \cdot \text{'})/6$$

By hand, we have $c_k = \langle \mathbf{v}^{(k)}, \mathbf{s} \rangle / 6$. Thus for example (note the complex conjugation!),

$$c_0 = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{7}{2}$$

$$c_1 = \frac{1}{6} \left(1 + (1 - j\sqrt{3}) + \frac{3}{2}(-1 - j\sqrt{3}) - 6 + \frac{5}{2}(-1 + j\sqrt{3}) + 2(1 + j\sqrt{3}) \right) = -1 + j\frac{\sqrt{3}}{3}$$

$$c_5 = \frac{1}{6} \left(1 + (1 + j\sqrt{3}) + \frac{3}{2}(-1 + j\sqrt{3}) - 6 + \frac{5}{2}(-1 - j\sqrt{3}) + 2(1 - j\sqrt{3}) \right) = -1 - j\frac{\sqrt{3}}{3}$$

8. An alternative—and equivalent—notation for the projection of \mathbf{s} onto the columns of \mathbf{V} involves the discrete Fourier transform \mathbf{S} , defined by

$$\mathbf{S} \stackrel{\text{def}}{=} \mathbf{V}^H \mathbf{s}$$

This vector consists of the N inner products $\langle \mathbf{v}^{(k)}, \mathbf{s} \rangle$, where $k = 0 : N - 1$. Since the index k represents frequency (the frequency $\omega_k = k(2\pi/N)$), the DFT vector \mathbf{S} is a signal in the so-called frequency domain. It is also known as the *spectrum* of \mathbf{s} .

9. The projection coefficients (in the previous notation) are given by

$$\mathbf{c} = \frac{1}{N} \mathbf{V}^H \mathbf{s} = \frac{1}{N} \mathbf{S}$$

Thus

$$\mathbf{s} = \mathbf{V} \mathbf{c} = \frac{1}{N} \mathbf{V} \mathbf{S}$$

The signal \mathbf{s} is also referred to as the inverse DFT of its spectrum \mathbf{S} .

10. The matrix \mathbf{V} is symmetric, since $V_{nk} = e^{j(2\pi/N)kn} = V_{kn}$. Therefore

$$\mathbf{V}^H = \mathbf{V}^* \stackrel{\text{def}}{=} \mathbf{W}$$

We will examine the Fourier matrix \mathbf{V} and its complex conjugate \mathbf{W} in detail later on.

11. **Example.** To understand the meaning of the DFT (or spectrum), suppose

$$\mathbf{s} \xleftrightarrow{\text{DFT}} \mathbf{S} = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$$

Both \mathbf{s} and \mathbf{S} are of size $N = 8$. Since \mathbf{S} has only one nonzero entry, \mathbf{s} can be formed using one column of \mathbf{V} only, namely the $k = 0^{\text{th}}$ column ($= \mathbf{v}^{(0)}$). This means that \mathbf{s} consists of a single sinusoidal component at frequency $\omega = 0 \cdot (2\pi/8) = 0$; that sinusoid is, of course, constant in time. Thus \mathbf{s} is a constant vector, and its value can be found from the synthesis equation:

$$\mathbf{s} = \frac{1}{8} \mathbf{V} \mathbf{S} = \frac{1}{8} \mathbf{v}^{(0)}$$

i.e.,

$$\mathbf{s} = \frac{1}{8} [1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]^T$$

Your task: Verify this result using the `IFFT` command in MATLAB.

12. Like any DFT pair, the pair $\mathbf{s} \longleftrightarrow \mathbf{S}$ in the above example can be also obtained in the reverse direction, i.e., starting from

$$\mathbf{s} = \frac{1}{8} [1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]^T = \frac{1}{8} \mathbf{v}^{(0)}$$

and using the analysis equation

$$\mathbf{S} = \mathbf{V}^H \mathbf{s}$$

The right-hand side computes the inner product of each column of \mathbf{V} with \mathbf{s} . Since \mathbf{s} is a scaling of the zeroth column of \mathbf{V} , and since the columns of \mathbf{V} are orthogonal, the result is

$$\mathbf{S} = \frac{1}{8} [8 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$$

Your task: Verify this calculation using the `FFT` command in MATLAB.

This example can be generalized to any DFT that has a single nonzero entry.

LECTURE 13

Topic: examples of the DFT and its inverse; DFT of a real-valued signal

Textbook References: sections 3.2, 3.3

Key Points:

- The N -point vector \mathbf{s} is a complex sinusoid of frequency $\omega = 2k\pi/N$ if and only its DFT (or spectrum) \mathbf{S} contains all zeros except for $S[k] \neq 0$.

- Linearity property of the DFT: $\mathbf{s} = \alpha\mathbf{x} + \beta\mathbf{y} \xleftrightarrow{\text{DFT}} \mathbf{S} = \alpha\mathbf{X} + \beta\mathbf{Y}$

- The DFT \mathbf{S} of a N -point real-valued signal \mathbf{s} exhibits circular conjugate symmetry:

$$S[0] = S^*[0] \quad \text{and} \quad S[N - k] = S^*[k], \quad k = 1 : N - 1 \quad (\text{DFT 2})$$

- Every real-valued signal \mathbf{s} can be expressed as

$$s[n] = \frac{1}{N}S[0] + \frac{2}{N} \cdot \sum_{0 < k < N/2} |S[k]| \cos\left(\frac{2\pi kn}{N} + \angle S[k]\right) + \frac{1}{N}S[N/2](-1)^n$$

The second term (corresponding to frequency $\omega = \pi$) is present only when N is even.

Theory and Examples:

1. In the previous lecture, we considered DFT vectors (or spectra) with a single nonzero entry. The corresponding time domain-signals are Fourier sinusoids. Thus:

- If

$$\mathbf{S} = [0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0]^T$$

then the time-domain signal \mathbf{s} is a sinusoid of frequency $\omega = 4(2\pi/8) = \pi$. Specifically,

$$\mathbf{s} = \frac{1}{8}\mathbf{v}^{(4)} = \frac{1}{8} [1 \ -1 \ 1 \ -1 \ 1 \ -1 \ 1 \ -1]^T$$

since

$$s[n] = \frac{1}{8}e^{j\pi n} = \frac{(-1)^n}{8}, \quad n = 0 : 7$$

- If

$$\mathbf{S} = [0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0]^T$$

then $\mathbf{s} = \mathbf{v}^{(3)}/8$. This is a sinusoid of frequency $\omega = 3(2\pi/8) = 3\pi/4$, and is given by

$$s[n] = \frac{1}{8}e^{j(3\pi/4)n}, \quad n = 0 : 7$$

In other words,

$$\mathbf{s} = \frac{1}{8} \left[1 \quad -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} \quad -j \quad \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} \quad -1 \quad \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} \quad j \quad -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} \right]^T$$

- We now introduce a second nonzero entry in \mathbf{S} , i.e., a second sinusoidal component in \mathbf{s} . Letting

$$\mathbf{S} = [0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0]^T$$

we have

$$\mathbf{s} = \frac{\mathbf{v}^{(3)} + \mathbf{v}^{(5)}}{8}$$

which is the scaled sum of two columns of \mathbf{V} . The corresponding frequencies are $\omega = 3\pi/4$ (as before); and $\omega = 5\pi/4$, which is the same frequency as $\omega = -3\pi/4$ for complex sinusoids. Since

$$e^{j\omega n} + e^{-j\omega n} = 2 \cos \omega n$$

it follows that

$$s[n] = \frac{1}{4} \cos(3\pi n/4), \quad n = 0 : 7$$

i.e.,

$$\mathbf{s} = \frac{1}{8} [2 \quad -\sqrt{2} \quad 0 \quad \sqrt{2} \quad -2 \quad \sqrt{2} \quad 0 \quad -\sqrt{2}]^T$$

Your task: Verify the results using the IFFT command in MATLAB.

2. The last example also demonstrates an important property of the DFT and its inverse, namely linearity. Since both the analysis and synthesis equations involve multiplication of a variable vector by a fixed matrix (\mathbf{W} or \mathbf{V}), we have

$$\mathbf{s} = \alpha \mathbf{x} + \beta \mathbf{y} \iff \mathbf{S} = \alpha \mathbf{X} + \beta \mathbf{Y} \quad (\text{DFT 1})$$

Based on this property and the DFT pairs obtained previously, we can also invert (for example)

$$\mathbf{S} = [1 \ 0 \ 0 \ 3 \ -2 \ 3 \ 0 \ 0]^T$$

The time-domain vector \mathbf{s} is a linear combination of the vectors obtained earlier:

$$s[n] = \frac{1}{8} + \frac{3}{4} \cos(3\pi n/4) - \frac{1}{4}(-1)^n, \quad n = 0 : 7$$

In MATLAB:

```
n = (0:7).';
s = 1/8 - (-1).^n/4 + 3*cos(3*pi*n/4)/4 ;
S = fft(s)
```

3. In the last example, the real sinusoidal component

$$\frac{3}{4} \cos(3\pi n/4)$$

was formed by adding together two complex-valued sinusoids, at frequencies

$$\omega = \frac{3\pi}{4} \quad \text{and} \quad \omega = \frac{5\pi}{4} \quad \left(\text{same as } -\frac{3\pi}{4} \right)$$

namely the third and fifth Fourier frequencies for an eight-point vector. The choice

$$S[3] = S[5]$$

is an instance of a certain type of symmetry about the “middle” frequency index $k = N/2 = 4$, which we will explore further. For the time being, note that a choice such as $S[3] = 3$ and $S[5] = -1$ would have resulted in $\mathbf{s} \neq \Re\{\mathbf{s}\}$. (**Question:** Why?)

4. As we will soon see, if a time-domain vector \mathbf{s} is real-valued, then:

- the entries of the DFT \mathbf{S} are not necessarily real; they are, however, symmetric (in a sense soon to be defined) about $k = N/2$.
- knowing \mathbf{S} , we can express \mathbf{s} as a sum of real-valued sinusoids at Fourier frequencies, all of which can be limited to the interval $[0, \pi]$.

5. The aforementioned symmetry is revealed using the analysis equation

$$\mathbf{S} = \mathbf{V}^H \mathbf{s}$$

where $S[k]$ is simply the inner product of the k^{th} Fourier sinusoid $\mathbf{v}^{(k)}$ and \mathbf{s} . The sinusoids $\mathbf{v}^{(k)}$ and $\mathbf{v}^{(N-k)}$ are complex conjugates of each other, since they are at frequencies

$$\omega = \frac{2\pi k}{N} \quad \text{and} \quad \omega = \frac{2\pi(N-k)}{N} = 2\pi - \frac{2\pi k}{N}$$

(recall that $(e^{j\theta})^* = e^{-j\theta}$). This means that $S[k]$ and $S[N-k]$ can be written as

$$\begin{aligned} S[k] &= (\mathbf{a} + j\mathbf{b})^T \mathbf{s} = \mathbf{a}^T \mathbf{s} + j\mathbf{b}^T \mathbf{s} \\ S[N-k] &= (\mathbf{a} - j\mathbf{b})^T \mathbf{s} = \mathbf{a}^T \mathbf{s} - j\mathbf{b}^T \mathbf{s} \end{aligned}$$

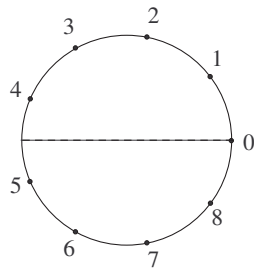
where \mathbf{a} and \mathbf{b} are real-valued vectors (consisting of cosines and sines, respectively). Since \mathbf{s} is real-valued, so are the dot products $\mathbf{a}^T \mathbf{s}$ and $\mathbf{b}^T \mathbf{s}$, and therefore

$$S[N-k] = S^*[k], \quad k = 1 : N-1$$

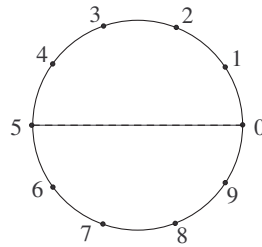
6. **Your Task:** Verify this property in MATLAB:

```
s1 = randn(9,1) ;
S1 = fft(s1)      % (one real value and four complex conjugate pairs)
s2 = randn(10,1) ;
S2 = fft(s2)      % (two real values and four complex conjugate pairs)
```

7. The property $S[N-k] = S^*[k]$ is known as *circular conjugate symmetry*. If the indices $k = 0 : N-1$ are marked counterclockwise and uniformly on the unit circle with $k = 0$ at angle $\theta = 0$, then positions k and $N-k$ are symmetric about the horizontal (real) axis.



$N = 9$



$N = 10$

8. The indices $k = 0$ and $k = N/2$ deserve special mention. The first index corresponds to the frequency $\omega = 0$. Accordingly, $S[0]$ is the inner product of the all-ones vector $\mathbf{v}^{(0)}$ with \mathbf{s} , and is clearly real-valued. Therefore

$$S[0] = S^*[0] = s[0] + s[1] + \cdots + s[N - 1]$$

The index $k = N/2$ arises only when N is even. The corresponding sinusoid $\mathbf{v}^{(N/2)}$ has frequency π , i.e., it is a real-valued vector consisting of alternating values 1 and -1 . Therefore $S[N/2]$ is also real valued, and

$$S[N/2] = S^*[N/2] = s[0] - s[1] + \cdots - s[N - 1]$$

9. Obviously, an arbitrary real-valued signal \mathbf{s} has a complex-valued spectrum (DFT) \mathbf{S} . This complex vector is often expressed in polar form, entry by entry:

$$S[k] = |S[k]| \cdot e^{j\angle S[k]}$$

This gives rise to two spectra:

- The amplitude, or magnitude, spectrum $|S[k]|$. Viewed as a vector, this is circularly symmetric in k :

$$S[N - k] = S^*[k] \quad \Rightarrow \quad |S[N - k]| = |S[k]|$$

- The phase, or angle, spectrum $\angle S[k]$. Viewed as a vector, this is circularly antisymmetric in k :

$$S[N - k] = S^*[k] \quad \Rightarrow \quad \angle S[N - k] = -\angle S[k]$$

In the special cases $k = 0$ and $k = N/2$, the value of $S[k]$ is real. Whenever $S[k]$ is real, $\angle S[k] = 0$ if $S[k] \geq 0$ and $\angle S[k] = \pi$ (equivalent to $-\pi$) if $S[k] < 0$.

10. We can now generalize the example given earlier in order to express an arbitrary real-vector of length $N = 8$ as a sum of real-valued sinusoids at Fourier frequencies 0, $\pi/4$, $\pi/2$, $3\pi/4$ and π . Suppose the DFT of \mathbf{s} is given by

$$\mathbf{S} = [1 \quad 1 + 2j \quad -4j \quad 3 - j \quad -2 \quad 3 + j \quad 4j \quad 1 - 2j]^T$$

The synthesis equation

$$\mathbf{s} = \frac{1}{8} \mathbf{V} \mathbf{S}$$

will again combine complex sinusoids with frequency indices k and $N - k$ into a single real-valued sinusoid. The coefficient of that sinusoid can be obtained by noting that if

$$S[k] = Ae^{j\phi} \quad \text{and} \quad S[N - k] = Ae^{-j\phi}$$

then the N entries of the vector

$$S[k] \mathbf{v}^{(k)} + S[N - k] \mathbf{v}^{(N-k)}$$

are given (for $n = 0 : N - 1$) by the formula

$$Ae^{j\phi} e^{j\omega n} + Ae^{-j\phi} e^{-j\omega n} = 2A \cos(\omega n + \phi)$$

where $\omega = k(2\pi/N)$.

The values of $A = |S[k]|$ and $\phi = \angle S[k]$ are found by expressing \mathbf{S} in polar form—i.e., in terms of the amplitude and phase spectra. The final result is

$$s[n] = \frac{1}{8} + \frac{\sqrt{5}}{4} \cos\left(\frac{\pi n}{4} + 1.1071\right) + \cos\left(\frac{\pi n}{2} - \frac{\pi}{2}\right) + \frac{\sqrt{10}}{4} \cos\left(\frac{3\pi n}{4} - 0.3218\right) - \frac{1}{4}(-1)^n$$

LECTURE 14

Topics: circular shift and reversal; circular conjugate symmetry of the DFT and IDFT matrices; implications of conjugation and circular reversal on the DFT

Textbook References: sections 3.2.2, 3.3.2, 3.4

Key Points:

- $\mathbf{P}\mathbf{x}$ is a circular shift (downward by one position) in the entries of \mathbf{x} . $\mathbf{R}\mathbf{x}$ is a circular reversal in the entries of \mathbf{x} . $\mathbf{F}^k\mathbf{x}$ is the result of multiplying \mathbf{x} by the k^{th} Fourier sinusoid, entry by entry.
- The columns and rows of \mathbf{V} and $\mathbf{W} = \mathbf{V}^*$ exhibit circular conjugate symmetry.
-

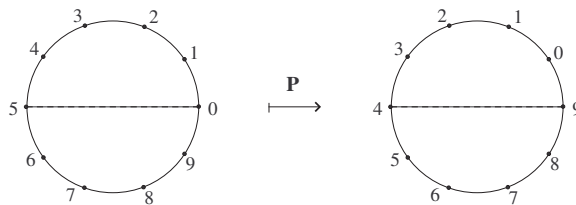
$$\begin{array}{ccc}
 \textit{Time Domain} & & \textit{Frequency Domain} \\
 \mathbf{x} = \frac{1}{N}\mathbf{V}\mathbf{X} & \longleftrightarrow & \mathbf{X} = \mathbf{W}\mathbf{x} \\
 \mathbf{x}^* & \longleftrightarrow & \mathbf{R}\mathbf{X}^* \\
 \mathbf{R}\mathbf{x} & \longleftrightarrow & \mathbf{R}\mathbf{X}
 \end{array}$$

Theory and Examples:

1. A *circular shift* of an N -vector is the entry permutation described by

$$[x[0] \quad x[1] \quad \dots \quad x[N-2] \quad x[N-1]]^T \mapsto [x[N-1] \quad x[0] \quad x[1] \quad \dots \quad x[N-2]]^T$$

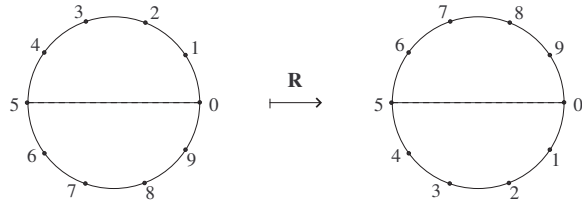
and illustrated below.



The associated permutation matrix \mathbf{P} (acting on the column vector \mathbf{x} to produce $\mathbf{P}\mathbf{x}$) is the $N \times N$ matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

\mathbf{P}^m (m th power of \mathbf{P}) represents a circular shift by m positions, and $\mathbf{P}^N = \mathbf{I}$.



2. A *circular reversal* is the entry permutation described by

$$[x[0] \quad x[1] \quad \dots \quad x[N-2] \quad x[N-1]]^T \mapsto [x[0] \quad x[N-1] \quad \dots \quad x[2] \quad x[1]]^T$$

and illustrated above.

The associated permutation matrix \mathbf{R} is

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \swarrow & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \end{bmatrix}$$

Clearly, $\mathbf{R}^2 = \mathbf{I}$.

3. The matrix \mathbf{F} is not a permutation matrix. It is a $N \times N$ diagonal matrix whose leading diagonal has the same entries as the $k = 1^{\text{st}}$ column of \mathbf{V} (the sinusoid of frequency $\omega = 2\pi/N$):

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & v & 0 & \dots & 0 \\ 0 & 0 & v^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & v^{N-1} \end{bmatrix},$$

where $v = e^{j(2\pi/N)}$.

4. \mathbf{F}^k is also a diagonal matrix. Its leading diagonal entries are given by the k^{th} column of \mathbf{V} . The effect of the transformation

$$\mathbf{y} = \mathbf{F}^k \mathbf{x}$$

is to multiply each entry of \mathbf{x} by the corresponding entry in the k^{th} Fourier sinusoid:

$$y[n] = e^{jk(2\pi/N)n} x[n], \quad n = 0 : N-1$$

Question: If N is even, how can you express the vector \mathbf{y} given (for all n) by $y[n] = (-1)^n x[n]$ in terms of \mathbf{F} and \mathbf{x} ?

Note that if $\mathbf{1}$ is the all-ones column vector (same as $\mathbf{v}^{(0)}$), then the time-domain signal \mathbf{s} and its DFT \mathbf{S} satisfy

$$S[k] = \mathbf{1}^T \mathbf{F}^{-k} \mathbf{s} \quad \text{and} \quad N s[n] = \mathbf{1}^T \mathbf{F}^n \mathbf{S}$$

5. **Example.** Express $\mathbf{x}^{(1)}$ through $\mathbf{x}^{(6)}$ (below) in terms of \mathbf{P} , \mathbf{R} , \mathbf{F} and

$$\mathbf{x} = [0 \ 1 \ 2 \ 3 \ 4 \ 5]^T$$

- $\mathbf{x}^{(1)} = [4 \ 5 \ 0 \ 1 \ 2 \ 3]^T$
- $\mathbf{x}^{(2)} = [4 \ 3 \ 2 \ 1 \ 0 \ 5]^T$
- $\mathbf{x}^{(3)} = [0 \ 6 \ 6 \ 6 \ 6 \ 6]^T$
- $\mathbf{x}^{(4)} = [0 \ -4 \ -2 \ 0 \ 2 \ 4]^T$
- $\mathbf{x}^{(5)} = [0 \ -1 \ 2 \ -3 \ 4 \ -5]^T$
- $\mathbf{x}^{(6)} = [0 \ 2 \ 0 \ 6 \ 0 \ 10]^T$

6. We have seen that the matrices \mathbf{V} and $\mathbf{W} = \mathbf{V}^*$ are both symmetric. Their entries are integer powers of

$$v = e^{j(2\pi/N)} = w^{-1} = w^*$$

There are only N distinct such powers (e.g., $v^0 = 1$ through v^{N-1}), which are complex numbers uniformly distributed around the unit circle. They all appear in the $k = 1^{\text{st}}$ column of \mathbf{V} , i.e., in $\mathbf{v}^{(1)} = [1 \ v \ v^2 \ \dots \ v^{N-1}]^T$.

7. As we noted in the previous reading assignment, the k^{th} and $(N-k)^{\text{th}}$ columns of \mathbf{V} (namely the sinusoids $\mathbf{v}^{(k)}$ and $\mathbf{v}^{(N-k)}$) are complex conjugates of each other. The zeroth column $\mathbf{v}^{(0)}$ is clearly real-valued. Thus

$$\mathbf{v}^{(0)} = (\mathbf{v}^{(0)})^* \quad \text{and} \quad \mathbf{v}^{(N-k)} = (\mathbf{v}^{(k)})^*, \quad k = 1 : N - 1$$

which means that the columns of \mathbf{V} have the same circular conjugate symmetry property as the DFT of a real-valued signal. The same is true for the rows of \mathbf{V} , since \mathbf{V} is symmetric.

8. The circular conjugate symmetry of the columns and rows of \mathbf{V} can be expressed in terms of the circular reversal operation defined earlier (which swaps pairs of indices k and $N-k$ while leaving index 0 in place). We have

$$\mathbf{V}^* = \mathbf{V}\mathbf{R}^T = \mathbf{R}\mathbf{V}$$

Since \mathbf{R} is symmetric and $\mathbf{V}^* = \mathbf{W}$, we can rewrite this as

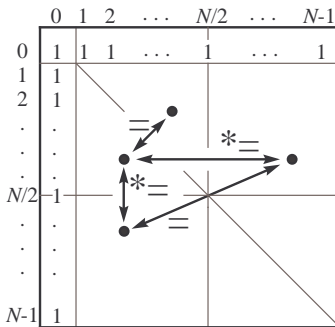
$$\mathbf{W} = \mathbf{V}^* = \mathbf{V}\mathbf{R} = \mathbf{R}\mathbf{V}$$

and taking complex conjugates of all terms, we obtain

$$\mathbf{V} = \mathbf{W}^* = \mathbf{W}\mathbf{R} = \mathbf{R}\mathbf{W}$$

In other words, the columns and rows of \mathbf{W} also exhibit circular conjugate symmetry.

9. The symmetry properties of \mathbf{V} and \mathbf{W} are illustrated below, where equal values are denoted by “=” and conjugate values by “*”. One of the equalities corresponds to $\mathbf{R}\mathbf{W}\mathbf{R} = \mathbf{W}$ and $\mathbf{R}\mathbf{V}\mathbf{R} = \mathbf{V}$. (**Your Task:** Prove both identities.)



10. Obviously, modifying a signal \mathbf{x} also changes its spectrum \mathbf{X} . Our focus will be on certain transformations of \mathbf{x} that result in \mathbf{X} changing in a well-structured manner.
11. **(DFT 3)** Taking the complex conjugate of \mathbf{x} results in both complex conjugation and circular reversal of the entries of \mathbf{X} .

This is easily shown using the circular conjugate symmetry of \mathbf{W} , i.e., $\mathbf{WR} = \mathbf{RW} = \mathbf{W}^*$. Thus if $\mathbf{y} = \mathbf{x}^*$, then

$$\begin{aligned}
 \mathbf{Y} &= \mathbf{W}\mathbf{y} \\
 &= \mathbf{W}\mathbf{x}^* \\
 &= \mathbf{R}\mathbf{W}^*\mathbf{x}^* \\
 &= \mathbf{R}\mathbf{X}^*
 \end{aligned}$$

12. **(DFT 4)** Circular reversal of \mathbf{x} results in circular reversal of \mathbf{X} .

Again, we use the circular conjugate symmetry of \mathbf{W} . If $\mathbf{y} = \mathbf{R}\mathbf{x}$, then

$$\begin{aligned}
 \mathbf{Y} &= \mathbf{W}\mathbf{R}\mathbf{x} \\
 &= \mathbf{R}\mathbf{W}\mathbf{x} \\
 &= \mathbf{R}\mathbf{X}
 \end{aligned}$$

13. **Example.** If we know that

$$[1 \quad j \quad -2 \quad 2j]^T \longleftrightarrow [-1+3j \quad 2 \quad -1-3j \quad 4]^T$$

then we immediately obtain

$$[1 \quad -j \quad -2 \quad -2j]^T \longleftrightarrow [-1-3j \quad 4 \quad -1+3j \quad 2]^T$$

$$[1 \quad 2j \quad -2 \quad j]^T \longleftrightarrow [-1+3j \quad 4 \quad -1-3j \quad 2]^T$$

LECTURE 15

Topics: signal transformations and the DFT; duality; circular symmetry revisited

Textbook References: sections 3.4, 3.5

Key Points:

$$\begin{array}{ccc}
 \underline{\textit{Time Domain}} & & \underline{\textit{Frequency Domain}} \\
 \mathbf{x} = \frac{1}{N} \mathbf{V} \mathbf{X} & \longleftrightarrow & \mathbf{X} = \mathbf{W} \mathbf{x} \\
 \mathbf{P}^m \mathbf{x} & \longleftrightarrow & \mathbf{F}^{-m} \mathbf{X} \\
 \mathbf{F}^m \mathbf{x} & \longleftrightarrow & \mathbf{P}^m \mathbf{X} \\
 \mathbf{X} & \longleftrightarrow & N \mathbf{R} \mathbf{x}
 \end{array}$$

- Every DFT pair has a dual: if $\mathbf{x} \longleftrightarrow \mathbf{X}$, then $\mathbf{X} \longleftrightarrow N \mathbf{R} \mathbf{x}$.
- Circular conjugate symmetry in either domain (time or frequency) implies exclusively real values in the other domain, and vice versa. If a signal is both real-valued and circularly symmetric, then so is its spectrum.

Theory and Examples:

1. Let us take a closer look at circular shifts of the rows and/or columns of \mathbf{V} and \mathbf{W} . The diagonal matrix \mathbf{F} (whose leading diagonal consists of the entries of $\mathbf{v}^{(1)}$) is implicitly involved here. If $\mathbf{1}$ is the all-ones column vector, then we can write

$$\mathbf{V} = [\mathbf{1} \quad \mathbf{F} \mathbf{1} \quad \mathbf{F}^2 \mathbf{1} \quad \dots \quad \mathbf{F}^{N-1} \mathbf{1}]$$

A circular shift on the columns of \mathbf{V} is obtained by right-multiplying it by $\mathbf{P}^T = \mathbf{P}^{-1}$, resulting in

$$\begin{aligned}
 \mathbf{V} \mathbf{P}^{-1} &= [\mathbf{F}^{N-1} \mathbf{1} \quad \mathbf{1} \quad \mathbf{F} \mathbf{1} \quad \dots \quad \mathbf{F}^{N-2} \mathbf{1}] \\
 &= [\mathbf{F}^{-1} \mathbf{1} \quad \mathbf{1} \quad \mathbf{F} \mathbf{1} \quad \dots \quad \mathbf{F}^{N-2} \mathbf{1}] = \mathbf{F}^{-1} \mathbf{V}
 \end{aligned}$$

Similarly, a circular shift in the rows of \mathbf{V} (which are given by $\mathbf{1}^T \mathbf{F}^n$) results in

$$\mathbf{P} \mathbf{V} = \mathbf{V} \mathbf{F}^{-1}$$

These results can be generalized to any integer (positive or negative) power m of \mathbf{P} :

$$\mathbf{V} \mathbf{P}^m = \mathbf{F}^m \mathbf{V}, \quad \mathbf{P}^m \mathbf{V} = \mathbf{V} \mathbf{F}^{-m}$$

and can be extended to \mathbf{W} by taking complex conjugates:

$$\mathbf{W} \mathbf{P}^m = \mathbf{F}^{-m} \mathbf{W}, \quad \mathbf{P}^m \mathbf{W} = \mathbf{W} \mathbf{F}^m$$

2. (**DFT 5**) If we circularly shift the entries \mathbf{x} by m positions (m positive or negative), then the entries of \mathbf{X} are multiplied by those of the $(N - m)^{\text{th}}$ Fourier sinusoid.

To see this, we let $\mathbf{y} = \mathbf{P}^m \mathbf{x}$. Using one of the identities derived above, we obtain

$$\begin{aligned} \mathbf{Y} &= \mathbf{W} \mathbf{P}^m \mathbf{x} \\ &= \mathbf{F}^{-m} \mathbf{W} \mathbf{x} \\ &= \mathbf{F}^{-m} \mathbf{X} \\ &= \mathbf{F}^{N-m} \mathbf{X} \end{aligned}$$

As it turns out, a similar (though not identical) property holds with the two domains (time and frequency) interchanged:

3. (**DFT 6**) If we multiply the entries of \mathbf{x} by those of the m^{th} Fourier sinusoid, then the entries of \mathbf{X} undergo a circular shift by m positions.

Indeed, if $\mathbf{y} = \mathbf{F}^m \mathbf{x}$, then

$$\begin{aligned} \mathbf{Y} &= \mathbf{W} \mathbf{F}^m \mathbf{x} \\ &= \mathbf{P}^m \mathbf{W} \mathbf{x} \\ &= \mathbf{P}^m \mathbf{X} \end{aligned}$$

4. **Example.** If

$$[a \ b \ c \ d \ e \ f]^T \longleftrightarrow [A \ B \ C \ D \ E \ F]^T$$

then

$$[d \ e \ f \ a \ b \ c]^T \longleftrightarrow [A \ -B \ C \ -D \ E \ -F]^T$$

$$[a \ -b \ c \ -d \ e \ -f]^T \longleftrightarrow [D \ E \ F \ A \ B \ C]^T$$

$$[2a \ b \ -c \ -2d \ -e \ f]^T \longleftrightarrow [F+B \ A+C \ B+D \ C+E \ D+F \ E+A]^T$$

Your task: Derive each of the three new pairs.

5. The similarity between properties **DFT 5** and **DFT 6**, both of which involve circular shift in one domain and multiplication by a Fourier sinusoid in the other, is an instance of time-frequency duality. Duality stems from the fact that the linear transformations involved in the analysis and synthesis equations differ only by a complex conjugate (since $\mathbf{W} = \mathbf{V}^*$) and a scaling factor $1/N$.
6. Every time we compute a DFT pair $\mathbf{x} \longleftrightarrow \mathbf{X}$, we also obtain a second DFT pair with very little additional algebra. The two pairs are known as duals of each other, and are characterized by the feature that the two vectors \mathbf{x} and \mathbf{X} are swapped between the two domains. In precise terms:

7. (DFT 7) If $\mathbf{x} \longleftrightarrow \mathbf{X}$, then

$$\mathbf{y} = \mathbf{X} \longleftrightarrow \mathbf{Y} = N\mathbf{R}\mathbf{x}$$

This can be proved by using the circular conjugate symmetry of \mathbf{W} :

$$\begin{aligned} \mathbf{Y} &= \mathbf{W}\mathbf{y} \\ &= \mathbf{W}\mathbf{X} \\ &= \mathbf{R}\mathbf{V}\mathbf{X} \\ &= N\mathbf{R}\mathbf{x} \end{aligned}$$

(Equivalently, $\mathbf{R}\mathbf{X} \longleftrightarrow N\mathbf{x}$.)

8. **Example.** If

$$[a \ b \ c \ d \ e \ f]^T \longleftrightarrow [3 \ 1-j \ 2+j \ -1 \ 2-j \ 1+j]^T$$

then

$$[3 \ 1-j \ 2+j \ -1 \ 2-j \ 1+j]^T \longleftrightarrow 6[a \ f \ e \ d \ c \ b]^T$$

9. We know (from DFT 2 or DFT 3) that

$$\mathbf{x} = \mathbf{x}^* \longleftrightarrow \mathbf{X} = \mathbf{R}\mathbf{X}^*$$

i.e., a real-valued signal has a circularly conjugate symmetric spectrum. As a dual property, we have that

$$\mathbf{x} = \mathbf{R}\mathbf{x}^* \longleftrightarrow \mathbf{X} = \mathbf{X}^*$$

Your task: Prove this.

10. If \mathbf{x} is both real and circularly symmetric, then so is its spectrum \mathbf{X} . In other words,

$$\mathbf{x} = \mathbf{x}^* = \mathbf{R}\mathbf{x} \longleftrightarrow \mathbf{X} = \mathbf{X}^* = \mathbf{R}\mathbf{X}$$

(The modifier “conjugate” is redundant in this case.)

11. **Example.** Verify in MATLAB that

$$\mathbf{x} = [-1 \ 3 \ 2 \ 1 \ 4 \ 1 \ 2 \ 3]^T$$

has a real and circularly symmetric DFT \mathbf{X} .

Question: If the signal $x[n]$ is expressed as a sum of five real-valued Fourier sinusoids, what special features will those sinusoids have?

12. In regard to the transformations \mathbf{R} , \mathbf{P} and \mathbf{F} introduced so far:

- If \mathbf{x} is real-valued, and/or if \mathbf{x} is circularly conjugate symmetric, then so is $\mathbf{R}\mathbf{x}$.
- If \mathbf{x} is real-valued, then so is $\mathbf{P}^m\mathbf{x}$ for any m . If \mathbf{x} is circularly conjugate symmetric, then so are $\mathbf{P}^{N/2}\mathbf{x}$ and $(\mathbf{P}^m + \mathbf{P}^{-m})\mathbf{x}$ for any m .
- If \mathbf{x} is circularly conjugate symmetric, then so is $\mathbf{F}^m\mathbf{x}$ for any m . If \mathbf{x} is real-valued, then so are $\mathbf{F}^{N/2}\mathbf{x}$ and $(\mathbf{F}^m + \mathbf{F}^{-m})\mathbf{x}$ for any m .

Your task: Prove these facts. Use the vector in the previous example and the functions OFLIP, OSHIFT and FDIAG (given in M-files) to illustrate these statements in MATLAB.

LECTURE 16

Topics: multiplication and circular convolution; periodic extension of a vector

Textbook References: sections 3.6, 3.7

Key Points:

- The circular convolution of two N -length vectors \mathbf{a} and \mathbf{b} is the N -length vector $\mathbf{a} \circledast \mathbf{b}$ whose n^{th} entry is given by $\mathbf{a}^T \mathbf{P}^n \mathbf{R} \mathbf{b}$.
- If $\mathbf{a} \diamond \mathbf{b}$ denotes the element-by-element product of \mathbf{a} and \mathbf{b} , then

$$\begin{aligned} \mathbf{x} \diamond \mathbf{y} &\longleftrightarrow \frac{1}{N} \mathbf{X} \circledast \mathbf{Y} \\ \mathbf{x} \circledast \mathbf{y} &\longleftrightarrow \mathbf{X} \diamond \mathbf{Y} \end{aligned}$$

- If \mathbf{x} is formed by concatenating M copies of the L -point vector \mathbf{s} , then the DFT \mathbf{X} is obtained by inserting $M - 1$ zeros between consecutive entries of \mathbf{S} , and scaling the result by M .

Theory and Examples:

1. The element-by-element product $\mathbf{x} \diamond \mathbf{y}$ of two N -point vectors \mathbf{x} and \mathbf{y} is defined by

$$(\mathbf{x} \diamond \mathbf{y})[n] = x[n]y[n], \quad n = 0 : N - 1$$

Its DFT can be obtained directly from \mathbf{X} and \mathbf{Y} using a procedure known as circular convolution.

2. If $\mathbf{s} = \mathbf{x} \diamond \mathbf{y}$, then for every frequency index k ,

$$S[k] = \sum_{n=0}^{N-1} x[n]y[n]v^{-kn}$$

The product $y[n]v^{-kn}$ is the n^{th} entry of the vector

$$\mathbf{y} \diamond \mathbf{v}^{(N-k)} = \mathbf{F}^{-k} \mathbf{y}$$

We can therefore write

$$S[k] = \mathbf{x}^T \mathbf{F}^{-k} \mathbf{y}$$

Using the synthesis equation; known facts about \mathbf{V} ; and the analysis equation, we obtain

$$\begin{aligned} S[k] &= \frac{1}{N} (\mathbf{V} \mathbf{X})^T \mathbf{F}^{-k} \mathbf{y} \\ &= \frac{1}{N} \mathbf{X}^T \mathbf{V} \mathbf{F}^{-k} \mathbf{y} \\ &= \frac{1}{N} \mathbf{X}^T \mathbf{P}^k \mathbf{V} \mathbf{y} \\ &= \frac{1}{N} \mathbf{X}^T \mathbf{P}^k \mathbf{R} \mathbf{W} \mathbf{y} \\ &= \frac{1}{N} \mathbf{X}^T \mathbf{P}^k \mathbf{R} \mathbf{Y} \end{aligned}$$

3. The computation of the dot product $\mathbf{a}^T \mathbf{P}^n \mathbf{Rb}$ for every index $n = 0 : N - 1$ is known as circular convolution. The result is the vector $\mathbf{a} \circledast \mathbf{b}$ defined by

$$(\mathbf{a} \circledast \mathbf{b})[n] = \mathbf{a}^T \mathbf{P}^n \mathbf{Rb}$$

Circular convolution is commutative, i.e., $\mathbf{a} \circledast \mathbf{b} = \mathbf{b} \circledast \mathbf{a}$. This follows directly from the fact that circular convolution in the frequency domain is equivalent to scaled multiplication in the time domain; multiplication is, of course, commutative.

Also: The entries of $\mathbf{a} \circledast \mathbf{b}$ are equivalently expressed as

$$(\mathbf{a} \circledast \mathbf{b})[n] = \sum_{m=0}^{N-1} a[m] \cdot b[(n - m) \text{ or } (n - m + N)], \quad n = 0, 1, \dots, N - 1$$

where the argument if \mathbf{b} is $n - m$ whenever $m \leq n$, and $n - m + N$ otherwise.

4. **Example.** Let $\mathbf{a} = [\kappa \ \lambda \ \mu \ \nu]^T$ and $\mathbf{b} = [1 \ 2 \ 3 \ -2]^T$. We then have

$$\begin{aligned} \mathbf{P}^0 \mathbf{Rb} &= [1 \ -2 \ 3 \ 2]^T \\ \mathbf{P}^1 \mathbf{Rb} &= [2 \ 1 \ -2 \ 3]^T \\ \mathbf{P}^2 \mathbf{Rb} &= [3 \ 2 \ 1 \ -2]^T \\ \mathbf{P}^3 \mathbf{Rb} &= [-2 \ 3 \ 2 \ 1]^T \end{aligned}$$

The four dot products $\mathbf{a}^T \mathbf{P}^n \mathbf{Rb}$ ($= (\mathbf{P}^n \mathbf{Rb})^T \mathbf{a}$) can be computed in a single matrix-vector product:

$$\begin{bmatrix} 1 & -2 & 3 & 2 \\ 2 & 1 & -2 & 3 \\ 3 & 2 & 1 & -2 \\ -2 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} \kappa \\ \lambda \\ \mu \\ \nu \end{bmatrix} = \begin{bmatrix} \kappa - 2\lambda + 3\mu + 2\nu \\ 2\kappa + \lambda - 2\mu + 3\nu \\ 3\kappa + 2\lambda + \mu - 2\nu \\ -2\kappa + 3\lambda + 2\mu + \nu \end{bmatrix} = \mathbf{a} \circledast \mathbf{b}$$

5. The DFT property

$$\mathbf{x} \diamond \mathbf{y} \longleftrightarrow \frac{1}{N} \mathbf{X} \circledast \mathbf{Y} \quad (\text{DFT 8})$$

established earlier also has a dual:

$$\mathbf{x} \circledast \mathbf{y} \longleftrightarrow \mathbf{X} \diamond \mathbf{Y} \quad (\text{DFT 9})$$

Thus circular convolution in the time domain is equivalent to element-by-element multiplication in the frequency domain.

6. We can derive **DFT 9** from **DFT 8** using the duality property **DFT 7**. The pairs

$$\mathbf{R}\mathbf{x} \longleftrightarrow \mathbf{R}\mathbf{X} \quad \text{and} \quad \mathbf{R}\mathbf{y} \longleftrightarrow \mathbf{R}\mathbf{Y}$$

have duals

$$\mathbf{R}\mathbf{X} \longleftrightarrow N\mathbf{x} \quad \text{and} \quad \mathbf{R}\mathbf{Y} \longleftrightarrow N\mathbf{y}$$

From **DFT 8**, we have

$$\mathbf{R}(\mathbf{X} \diamond \mathbf{Y}) = \mathbf{R}\mathbf{X} \diamond \mathbf{R}\mathbf{Y} \quad \longleftrightarrow \quad \frac{N^2}{N} \cdot (\mathbf{x} \otimes \mathbf{y})$$

Applying **DFT 7** again, we obtain the required result:

$$N(\mathbf{x} \otimes \mathbf{y}) \quad \longleftrightarrow \quad N(\mathbf{X} \diamond \mathbf{Y})$$

7. Example. We circularly convolve the six-point vectors

$$\mathbf{a} = [1 \quad -2t \quad 3t^2 \quad 0 \quad 0 \quad 0]^T \quad \text{and} \quad \mathbf{b} = [2 \quad t \quad -t^2 \quad 2t^3 \quad 0 \quad 0]^T$$

where the n^{th} entry of each vector is a multiple of t^n . As before, we have

$$\mathbf{a} \otimes \mathbf{b} = \begin{bmatrix} 1 & 0 & 0 & 0 & 3t^2 & -2t \\ -2t & 1 & 0 & 0 & 0 & 3t^2 \\ 3t^2 & -2t & 1 & 0 & 0 & 0 \\ 0 & 3t^2 & -2t & 1 & 0 & 0 \\ 0 & 0 & 3t^2 & -2t & 1 & 0 \\ 0 & 0 & 0 & 3t^2 & -2t & 1 \end{bmatrix} \begin{bmatrix} 2 \\ t \\ -t^2 \\ 2t^3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3t \\ 3t^2 \\ 7t^3 \\ -7t^4 \\ 6t^5 \end{bmatrix}$$

The result $\mathbf{a} \otimes \mathbf{b}$ has the same distinctive property as \mathbf{a} and \mathbf{b} , namely the n^{th} entry is a multiple of t^n . Furthermore, the polynomials obtained from \mathbf{a} , \mathbf{b} and $\mathbf{a} \otimes \mathbf{b}$ by summing the entries of each vector together satisfy

$$(1 - 2t + 3t^2)(2 + t - t^2 + 2t^3) = 2 - 3t + 3t^2 + 7t^3 - 7t^4 + 6t^5$$

This result is closely linked to **DFT 9**. In fact, the three time-domain vectors \mathbf{x} , \mathbf{y} and $\mathbf{x} \otimes \mathbf{y}$ obtained by setting $t = 1$ in each of \mathbf{a} , \mathbf{b} and $\mathbf{a} \otimes \mathbf{b}$ have DFT's given by the three polynomials shown above, evaluated for $t = v^{-k}$, where $k = 0 : N - 1$.

8. Polynomial multiplication in MATLAB:

```
p = [ 1 -2 3 ].';
q = [ 2 1 -1 2 ].';
r = conv(p,q)
```

Circular convolution using **DFT 9**:

```
x = [ 1 -2 3 0 0 0 ].';
y = [ 2 1 -1 2 0 0 ].';
X = fft(x);
Y = fft(y);
S = X.*Y;
s = ifft(S)
```

The vectors \mathbf{r} and \mathbf{s} are identical.

9. The synthesis equation

$$s[n] = \frac{1}{L} \sum_{k=0}^{L-1} S[k] e^{j(2\pi/L)kn}, \quad n = 0 : L-1$$

expresses the L -point signal vector \mathbf{s} as weighted sum of L Fourier sinusoids over the index range $n = 0 : L-1$. Outside this range, each of these sinusoids repeats itself periodically every L samples. Thus if we extend the same equation to all integers n , i.e.,

$$p[n] = \frac{1}{L} \sum_{k=0}^{L-1} S[k] e^{j(2\pi/L)kn}, \quad n \in \mathbf{Z},$$

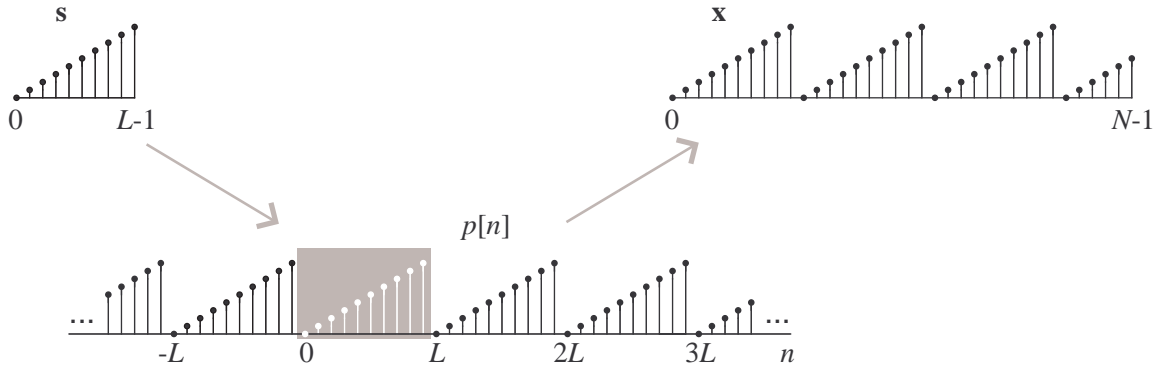
we obtain an infinite periodic sequence of period L , such that

$$\mathbf{s} = s[0 : L-1] = p[0 : L-1]$$

The sequence $p[\cdot]$ is a two-sided infinite periodic extension of the vector \mathbf{s} .

10. For the time being, our focus will be on one-sided finite (N -point) periodic extensions of \mathbf{s} , as illustrated below. In other words,

$$\mathbf{x} = x[0 : N-1] = p[0 : N-1]$$



11. The vector \mathbf{x} is given by the same equation, i.e.,

$$x[n] = \frac{1}{L} \sum_{k=0}^{L-1} S[k] e^{jk \frac{2\pi}{L} n}, \quad n = 0 : N-1 \quad (\spadesuit)$$

This is not, in general, a valid synthesis equation since the frequencies involved are not necessarily Fourier frequencies for vectors of length N . Thus the DFT \mathbf{X} is not immediately obtained from \mathbf{S} .

The case $N = ML$ (i.e., where extension is by a whole number of periods) is the only exception:

$$k^{\text{th}} \text{ frequency for length } L = k \frac{2\pi}{L} = kM \frac{2\pi}{ML} = (kM)^{\text{th}} \text{ frequency for length } ML$$

12. This means that in the case $N = ML$, the signal \mathbf{x} is a sum of only L (out of ML possible) sinusoids, and thus its spectrum has only L nonzero entries. The synthesis equation for $x[n]$ is

$$x[n] = \frac{1}{ML} \sum_{k=0}^{ML-1} X[k] e^{j(2\pi/(ML))kn}, \quad n = 0 : ML - 1$$

Comparing this with (\spadesuit), we deduce that

$$\mathbf{X} = M \cdot [S[0] \quad \mathbf{0}_{M-1}^T \quad S[1] \quad \mathbf{0}_{M-1}^T \quad \dots \quad S[L-1] \quad \mathbf{0}_{M-1}^T]^T$$

where $\mathbf{0}_n$ denotes a column vector of n zeros.

13. **Example.** Here $L = 3$, $M = 4$. If

$$\mathbf{s} = [a \quad b \quad c]^T \quad \longleftrightarrow \quad \mathbf{S} = [A \quad B \quad C]^T,$$

then

$$\mathbf{x} = [a \quad b \quad c \quad a \quad b \quad c \quad a \quad b \quad c \quad a \quad b \quad c]^T$$

has DFT

$$\mathbf{X} = [4A \quad 0 \quad 0 \quad 0 \quad 4B \quad 0 \quad 0 \quad 0 \quad 4C \quad 0 \quad 0 \quad 0]^T$$

LECTURE 17

Topics: zero-padded extension of a vector; detection of sinusoids using the DFT

Textbook References: sections 3.7, 3.8

Key Points:

- If \mathbf{y} is formed by appending $(M - 1)L$ zeros to the L -point vector \mathbf{s} , then the DFT \mathbf{Y} will contain the entries of \mathbf{S} at positions $0, M, \dots, (L - 1)M$.
- The frequency ω_0 of a discrete-time complex sinusoidal vector

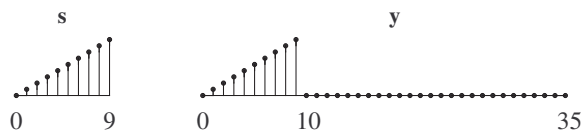
$$s[n] = e^{j\omega_0 n}, \quad n = 0 : L - 1$$

can be determined within $2\pi/N$ radians/sample, with $N > L$, by zero-padding \mathbf{s} to length N and locating the maximum in the resulting amplitude spectrum.

- The same technique can be used to estimate the frequency of a real-valued sinusoidal vector reliably. It can be also applied to a noisy sum of sinusoids provided a sufficiently large number of samples is taken.

Theory and Examples:

1. Another extension of interest is zero-padding, i.e., appending $N - L$ zeros to \mathbf{s} to obtain a vector \mathbf{y} of length N , as illustrated below.



There is no easy way of obtaining the DFT vector \mathbf{Y} from \mathbf{S} . However, in the special case $N = ML$, it is possible to proceed in the reverse direction—i.e., obtain \mathbf{S} from \mathbf{Y} .

2. Again, the (kM) th Fourier sinusoid for length ML is the same as the k th one for length L , only periodically repeated. This periodic repetition has no effect on the inner product with \mathbf{y} , since the nonzero portion of \mathbf{y} is limited to indices $0 : L - 1$, and is identical to \mathbf{s} . This means that, with $N = ML$, the two inner products

$$Y[kM] = \sum_{n=0}^{N-1} y[n]e^{-jkM\frac{2\pi}{N}n} = \sum_{n=0}^{L-1} y[n]e^{-jk\frac{2\pi}{L}n}$$

and

$$S[k] = \sum_{n=0}^{L-1} s[n]e^{-jk\frac{2\pi}{L}n}$$

are sums of the same nonzero terms, and are therefore equal.

Thus when $N = ML$, we can obtain \mathbf{S} by taking every M th entry in \mathbf{Y} .

3. **Example.** If ($L = 3, N = 12$)

$$\mathbf{y} = [a \ b \ c \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$$

has DFT

$$\mathbf{Y} = [Y_0 \ Y_1 \ Y_2 \ Y_3 \ Y_4 \ Y_5 \ Y_6 \ Y_7 \ Y_8 \ Y_9 \ Y_{10} \ Y_{11}]^T ,$$

then

$$[a \ b \ c]^T \longleftrightarrow [Y_0 \ Y_4 \ Y_8]^T$$

and

$$[a \ b \ c \ 0]^T \longleftrightarrow [Y_0 \ Y_3 \ Y_6 \ Y_9]^T$$

4. Appending a large number of zeros (so that $N \gg L$) results in a very dense set of Fourier frequencies in $[0, 2\pi)$, and a spectrum \mathbf{Y} which appears nearly continuous when plotted (e.g., as an amplitude spectrum and a phase spectrum) against ω . Thus zero-padding in the time domain is a way of interpolating in the frequency domain.
5. An important application of the DFT is in the efficient detection of sinusoidal and other periodic signals, often in the presence of noise. What makes the DFT a natural tool for this task is the fact that it analyzes a discrete signal vector into a number of sinusoidal components.
6. The equation

$$s(t) = A \cos(\Omega_0 t + \phi) + \epsilon(t) , \quad t \in \mathbf{R}$$

represents a single continuous-time sinusoid with additive noise $\epsilon(t)$. If we take L samples of $s(t)$ at times $t = 0 : T_s : (L - 1)T_s$, we obtain the vector \mathbf{s} given by

$$s[n] = A \cos(\omega_0 n + \phi) + \epsilon[n] , \quad n = 0 : L - 1$$

where

$$\omega_0 = \Omega_0 T_s = 2\pi \cdot \frac{f_0}{f_s} = 2\pi \cdot \frac{T_s}{T_0}$$

We are interested in estimating Ω_0 , A and ϕ using the samples in \mathbf{s} . The DFT \mathbf{S} , and more generally, the DFT of a zero-padded version of \mathbf{s} , can be used for that purpose.

7. The simplest case arises when the frequency ω_0 happens to be a Fourier frequency for the L -point sample \mathbf{s} . This occurs when

$$\omega_0 = k_0 \frac{2\pi}{L} \quad \Leftrightarrow \quad \frac{f_0}{f_s} = \frac{T_s}{T_0} = \frac{k_0}{L}$$

for some integer k_0 . The DFT \mathbf{S} will then be the sum of two vectors:

- The spectrum of $A \cos(\omega_0 n + \phi)$. Since ω_0 is the k_0^{th} Fourier frequency, this spectrum will have nonzero entries at $k = k_0$ and $k = L - k_0$ only, equal to

$$\frac{LA}{2} \cdot e^{j\phi} \quad \text{and} \quad \frac{LA}{2} \cdot e^{-j\phi} ,$$

respectively. (**Question:** Where have we seen this before?)

- The spectrum of the noise ϵ , which will have random entries spread over the L frequency indices.

If the noise is small compared to the sinusoidal signal (large ‘signal-to-noise ratio’) the amplitude spectrum (modulus of \mathbf{S}) will have two prominent peaks at frequency indices k_0 and $L - k_0$, from which ω_0 can be deduced. (**Question:** Why is aliasing a concern here?)

Estimates of A and ϕ can be obtained from the amplitude and phase spectra.

8. **Example.** The DFT was introduced earlier using a 200-point vector \mathbf{s} consisting of three sinusoids plus noise. The frequencies of the sinusoids coincided with the 12th, 19th and 23rd Fourier frequencies.

```
n = (0:199).';
s = 4.7*cos(0.12*pi*n-1.3) + ...
    3.8*cos(0.19*pi*n+0.8) + ...
    5.1*cos(0.23*pi*n+2.4) + ...
    2.0*randn(size(n)) ;
S = fft(s) ;
bar(n,abs(S)) , grid
```

Note the three prominent peaks on each half of the amplitude spectrum, at the correct frequency indices.

9. In almost all practical situations, ω_0 will not be an exact Fourier frequency for the L -point sample vector \mathbf{s} . This means that even though \mathbf{s} has a single dominant sinusoidal component, its representation in terms of Fourier sinusoids will contain significant amounts of all (Fourier) frequencies. Naturally, frequencies closer to ω_0 and $2\pi - \omega_0$ will be more prominent.

Your task: In the previous example, remove the noise (i.e., the fourth term in \mathbf{s}) and replace 0.12 by 0.12321, 0.19 by 0.19642 and 0.23 by 0.23419. Run again, and note how the amplitude spectrum changes.

10. To see how the DFT behaves when ω_0 is not a Fourier frequency, it is best to consider samples of a complex sinusoid, i.e.,

$$s[n] = e^{j\omega_0 n}, \quad n = 0 : L - 1$$

As usual, the DFT \mathbf{S} consists of the L inner products

$$\langle \mathbf{v}^{(\omega)}, \mathbf{s} \rangle = \sum_{n=0}^{L-1} s[n] e^{-jk \frac{2\pi}{L} n}$$

for $k = 0 : L - 1$. We will use the notation $\mathbf{v}^{(\omega)}$ to represent a L -point complex sinusoid of frequency ω (in the expression above, $\omega = 2\pi k/L$).

The inner product $\langle \mathbf{v}^{(\omega)}, \mathbf{s} \rangle$ can be also computed for a denser set of frequencies ω in $[0, 2\pi)$, namely all multiples of $2\pi/N$, where $N > L$. This is done by zero-padding \mathbf{s} to length N and computing a DFT. Taking $N \gg L$ and plotting the resulting amplitude spectrum against ω shows the behavior of $|\langle \mathbf{v}^{(\omega)}, \mathbf{s} \rangle|$ in great detail. Most notably, it shows that this quantity is maximized at $\omega = \omega_0$.

11. **Example.** Here $L = 20$ and $\omega_0 = (0.317)(2\pi)$. First, the DFT of \mathbf{s} is computed and plotted against $\omega/(2\pi)$. Then, \mathbf{s} is zero-padded to $N=1,000$ and another DFT is computed.

```
n = (0:19).';
f1 = (0 : 0.05 : 0.95).';
f2 = (0 : 0.001 : 0.999).';
s = exp(j*0.317*2*pi*n).';
stem( f1, abs(fft(s)), 'k' );
hold
plot( f2, abs(fft(s,1000)) );
```

Note that the larger N , the denser the Fourier frequencies, and hence the closer ω_0 is to a Fourier frequency, and the narrower the peak will be.

12. In short, the frequency of a noiseless complex sinusoid can be determined with arbitrary precision by zero-padding the signal and locating the maximum of the resulting amplitude spectrum. The fact that the maximum is achieved very close to ω_0 can be also shown theoretically by noting that

$$\langle \mathbf{v}^{(\omega)}, \mathbf{s} \rangle = \sum_{n=0}^{L-1} e^{j\omega_0 n} e^{-j\omega n} = \sum_{n=0}^{L-1} e^{j(\omega_0 - \omega)n}$$

is a geometric sum with common ratio $z = e^{j(\omega_0 - \omega)}$. By the usual formula,

$$\langle \mathbf{v}^{(\omega)}, \mathbf{s} \rangle = \frac{1 - z^L}{1 - z} = \frac{1 - e^{jL(\omega_0 - \omega)}}{1 - e^{j(\omega_0 - \omega)}}$$

which, after factoring out $z^{L/2}$ and $z^{1/2}$ from the numerator and denominator (respectively), reduces to

$$\langle \mathbf{v}^{(\omega)}, \mathbf{s} \rangle = e^{-j(L-1)(\omega - \omega_0)/2} \cdot \frac{\sin(L(\omega - \omega_0)/2)}{\sin((\omega - \omega_0)/2)}$$

The complex factor in the expression above has unit modulus, therefore

$$|\langle \mathbf{v}^{(\omega)}, \mathbf{s} \rangle| = \left| \frac{\sin(L(\omega - \omega_0)/2)}{\sin((\omega - \omega_0)/2)} \right|$$

A graph of this function of ω for $L = 20$ and $\omega_0 = (0.317)(2\pi)$ was obtained in the previous example. The maximum value of this function is L , and occurs at $\omega = \omega_0$.

LECTURE 18

Topics: periodic signals in continuous time; sums of harmonically related sinusoids; introduction to Fourier series

Key Points:

- Most periodic signals in continuous time can be expressed as sums of Fourier sinusoids.
- For a fundamental period T_0 and (correspondingly) fundamental frequency $\Omega_0 = 2\pi/T_0$, the k^{th} complex Fourier sinusoid (where $k \in \mathbf{Z}$) is given by

$$v^{(k)}(t) = e^{jk\Omega_0 t}$$

The frequency $k\Omega_0$ is referred to as the k^{th} harmonic (of Ω_0).

- The complex Fourier series of period T_0 with coefficients $\{S_k\}$ is defined by

$$s(t) = \sum_{k=-\infty}^{\infty} S_k e^{jk\Omega_0 t},$$

provided the infinite sum converges. It is real-valued if and only if

$$(\forall k \in \mathbf{Z}) \quad S_{-k} = (S_k)^*$$

Theory and Examples:

1. We have seen that the DFT synthesis equation

$$s[n] = \frac{1}{L} \sum_{k=0}^{L-1} S[k] e^{jk(2\pi/L)n}, \quad n = 0 : L - 1$$

can be extended outside the time index range $n = 0 : L - 1$ to produce an infinite periodic sequence having period L . This also means that *every* discrete-time sequence that is periodic with period L can be expressed as a sum of L Fourier sinusoids with coefficients given by the (scaled) DFT of the segment of the sequence corresponding to time indices $n = 0 : L - 1$.

Not surprisingly, the L Fourier frequencies $\omega_k = k(2\pi/L)$ (where $k = 0 : L - 1$) are the only distinct values of ω in $[0, 2\pi)$ such that $x[n] = e^{j\omega n}$ is periodic with period L .

2. *Most* periodic signals encountered in continuous time have a similar structure. In other words, they can be written as a sum of continuous-time Fourier sinusoids, whose coefficients are obtained by an operation that resembles the DFT.
3. Recall that a continuous-time signal $\{s(t), t \in \mathbf{R}\}$ is periodic with period $T_0 > 0$ if

$$(\forall t \in \mathbf{R}) \quad s(t + T_0) = s(t)$$

If $s(t)$ is periodic with period T_0 , it is also periodic with period mT_0 , where m is a positive integer. The *smallest* value of T_0 satisfying the above identity is the *fundamental period* of the signal.

Associated with any period T_0 are two familiar parameters: the cyclic frequency $f_0 = 1/T_0$ (in Hz) and the angular frequency (or simply frequency) $\Omega_0 = 2\pi/T_0$ (in rad/sec).

4. By analogy to discrete time, the continuous-time Fourier sinusoids corresponding to period T_0 are the signals

$$v^{(k)}(t) = e^{j\Omega_k t}$$

where

$$\Omega_k = k \cdot \frac{2\pi}{T_0} = k\Omega_0, \quad (k \in \mathbf{Z})$$

They are the only complex sinusoids of the form $e^{j\Omega t}$ which are periodic with period T_0 , i.e., have a fundamental period which is a submultiple of T_0 .

The fundamental frequency is Ω_0 , and the positive multiples $2\Omega_0, 3\Omega_0, \dots$ of Ω_0 are known as the *harmonics* of Ω_0 .

5. **Example.** Let $f_1 = 50.0$ Hz and $f_2 = 87.5$ Hz. Since the ratio f_2/f_1 is rational ($= 7/4$), f_1 and f_2 are both harmonics of the same fundamental cyclic frequency f_0 (another way of saying this is that they are *harmonically related*). The largest such value of f_0 is 12.5 Hz. (**Question:** What other values can f_0 take?)
6. Clearly, the Fourier frequencies $\{k\Omega_0, k \in \mathbf{Z}\}$ are a discrete, infinite set. Taking a linear combination of Fourier sinusoids results in the periodic signal

$$s(t) = \sum_{k=-\infty}^{\infty} S_k e^{jk\Omega_0 t} \quad (\spadesuit)$$

This is known as a *Fourier series of period T_0* with (complex-valued) coefficients $\{S_k, k \in \mathbf{Z}\}$. Depending on the values of the coefficients, the *fundamental* period of this signal may actually be a submultiple of T_0 .

Question: Why is $s(t)$ periodic with period T_0 ? When is the fundamental period of $s(t)$ equal to $T_0/2$?

Note that, unless only a finite number of S_k 's are nonzero, a Fourier series is an *infinite sum which may not converge*.

7. **Example.** Based on the previous example, the signal

$$s(t) = 5.0 + 6.2 \cos(25\pi t + 0.3) + 4.6 \cos(100\pi t - 1.9) + 2.8 \cos(175\pi t + 2.5)$$

is a Fourier series with fundamental period $T_0 = 1/(12.5) = 0.08$ sec. We can rewrite it in the complex form (\spadesuit) using the familiar formula

$$A \cos(\Omega t + \phi) = \frac{Ae^{j\phi}}{2} \cdot e^{j\Omega t} + \frac{Ae^{-j\phi}}{2} \cdot e^{-j\Omega t}$$

Thus the nonzero coefficients in (\spadesuit) are

$$\begin{aligned} S_0 &= 5.0 \\ S_1 &= (S_{-1})^* = 3.1e^{j0.3} \\ S_4 &= (S_{-4})^* = 2.3e^{-j1.9} \\ S_7 &= (S_{-7})^* = 1.4e^{j2.5} \end{aligned}$$

This signal is an example of a finite, or bandlimited, Fourier series; this means that the number of terms in the series is finite, and thus the frequencies present in $s(t)$ are all contained in a finite band (in this case, $[0, 87.5]$ Hz on the positive frequency axis).

8. In MATLAB, the Fourier series $s(t)$ of the previous example can be computed and plotted for an arbitrary (finite) set values of t . For example,

```
To = 0.08;                % fundamental period
N = 500;                  % samples per period
t = To*(0 : 1/N : 3-1/N).'; % three periods shown
s = 5.0 + 6.2*cos(25*pi*t+0.3) + ...
    4.6*cos(100*pi*t-1.9) + 2.8*cos(175*pi*t+2.5);
plot(t,s), grid, axis tight
```

An equivalent way of computing N uniformly spaced samples of $s(t)$ per period uses the DFT:

```
C = zeros(500,1);
C(1) = 5.0;
C(2) = 6.2*exp(j*0.3);
C(5) = 4.6*exp(-j*1.9);
C(8) = 2.8*exp(j*2.5);
c = N*ifft(C);           % N and t defined as above
c = real(c);
plot(t,[c;c;c]), grid, axis tight
```

The equivalence is due to the fact that, when N uniform samples of $s(t)$ are taken per period, each harmonic (sinusoidal) component of $s(t)$ becomes a (discrete-time) Fourier sinusoid for sample size N . **Your task:** Explore this further (in a homework assignment).

9. We now turn to an example of an infinite Fourier series, given in real form:

$$s(t) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{\pi k} \cdot \sin(k\Omega_0 t) \quad (\heartsuit)$$

In this case, $S_0 = 0$, while for $k \neq 0$,

$$S_k = j \cdot \frac{(-1)^k}{\pi k}$$

As we expect, $S_{-k} = (S_k)^*$ since $s(t)$ is real-valued.

10. The following iteration computes successive approximations to $s(t)$ using a finite sum (\sum_{-K}^K , with K increasing):

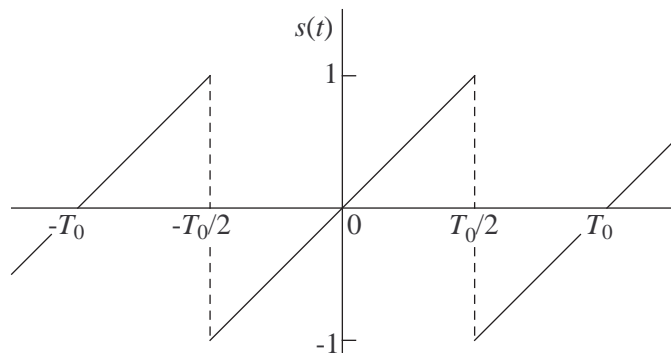
```
To = 0.08; N = 500;
t = To*(-1 : 1/N : 1-1/N).'; % two periods shown
s = 0*t;
```

```

for K = 1:100
    s = s + (2/pi)*(-1).^(K+1)*sin(2*pi*K*t/To)/K;
    plot(t,s), title(['K = ' num2str(K)])
    pause
end

```

11. It thus appears that as K increases, the finite sum (consisting of $2K$ complex sinusoids or, equivalently, K real sinusoids) approximates the “sawtooth” waveform shown below. The approximation is not uniformly good; pronounced ripples and overshoot are observed near the points of discontinuity.



It can be shown mathematically that as $K \rightarrow \infty$, the finite sum computed above converges at each t (except at points of discontinuity) to the ordinate obtained from the graph above. In other words, the infinite Fourier series (\heartsuit) is exactly equal the function shown above, with a minor adjustment at points of discontinuity:

$$s(t) = \begin{cases} 2t/T_0, & -T_0/2 < t < T_0/2; \\ 0, & t = T_0/2. \end{cases}$$

This formula is periodically extended outside $(-T_0/2, T_0/2]$.

12. As it turns out, most periodic signals in continuous time can be expressed (or “expanded”) as a Fourier series, i.e., they are a linear combination of harmonically related sinusoids. The infinite sum (\spadesuit) is the synthesis equation for this representation. The analysis equation, whereby the coefficients $\{S_k\}$ are obtained from the time-domain form of the signal $s(t)$, will be developed in the following lecture. Not surprisingly (i.e., by analogy to the DFT), the analysis equation involves an inner product in the time domain. But since the time domain is continuous, that inner product cannot be sum; it has to be an integral.

LECTURE 19

Topics: orthogonality of Fourier sinusoids; determination of Fourier series coefficients

Key Points:

- The inner product of two continuous-time periodic signals $f(t)$ and $g(t)$ with common period T_0 is defined by

$$\langle \mathbf{f}, \mathbf{g} \rangle = \frac{1}{T_0} \int_0^{T_0} f^*(t)g(t)dt$$

- The Fourier sinusoids $v^{(k)}(t) = e^{jk\Omega_0 t}$, where $k \in \mathbf{Z}$, are pairwise orthogonal and have unit norm:

$$\langle \mathbf{v}^{(k)}, \mathbf{v}^{(\ell)} \rangle = \begin{cases} 1, & k = \ell; \\ 0, & k \neq \ell. \end{cases}$$

- The coefficients of the Fourier series

$$s(t) = \sum_{k=-\infty}^{\infty} S_k e^{jk\Omega_0 t}$$

can be obtained from

$$S_k = \langle \mathbf{v}^{(k)}, \mathbf{s} \rangle = \frac{1}{T_0} \int_0^{T_0} s(t) e^{-jk\Omega_0 t} dt$$

Theory and Examples:

1. Most “nice” periodic signals in continuous time have a Fourier series expansion, i.e., they can be expressed as

$$s(t) = \sum_{k=-\infty}^{\infty} S_k e^{jk\Omega_0 t} \quad (\spadesuit)$$

where Ω_0 is the fundamental frequency. The proof of this fact—including important details such as the class of signals for which this representation holds and the manner in which the infinite sum converges—is beyond the scope of this course.

A more accessible, and also practical, question is the following: given a signal $s(t)$ which has the Fourier series representation shown above, how can we obtain the coefficients $\{S_k\}$? Put differently, if (\spadesuit) is the synthesis equation which constructs the signal $s(t)$ as a sum of Fourier sinusoids with coefficients $\{S_k\}$, what analysis equation produces each S_k from $s(t)$? As we will soon see, the answer involves the inner product of the k^{th} Fourier sinusoid and the signal $s(t)$.

2. Let $f(t)$ and $g(t)$ be complex-valued signals which are both periodic with period T_0 . Their inner product $\langle \mathbf{f}, \mathbf{g} \rangle$ is defined as

$$\langle \mathbf{f}, \mathbf{g} \rangle \stackrel{\text{def}}{=} \frac{1}{T_0} \int_0^{T_0} f^*(t)g(t)dt$$

We use boldface symbols to emphasize that, as was the case with vectors earlier, the inner product involves entire segments of the signals $f(t)$ and $g(t)$ (over an interval of length T_0).

3. The familiar properties

$$\begin{aligned}\langle \mathbf{g}, \mathbf{f} \rangle &= \langle \mathbf{f}, \mathbf{g} \rangle^* \\ \langle \mathbf{f}, z\mathbf{g} \rangle &= z\langle \mathbf{f}, \mathbf{g} \rangle \quad (z \in \mathbf{C}) \\ \langle \mathbf{f} + \mathbf{g}, \mathbf{h} \rangle &= \langle \mathbf{f}, \mathbf{h} \rangle + \langle \mathbf{g}, \mathbf{h} \rangle\end{aligned}$$

also hold in this case. We should further note that:

- The definition of $\langle \mathbf{f}, \mathbf{g} \rangle$ assumes that $f(t)$ and $g(t)$ are harmonically related, but the two signals need not have the same fundamental period. Thus T_0 can be any integer multiple of the fundamental period (of each signal); and clearly the same value is obtained for $\langle \mathbf{f}, \mathbf{g} \rangle$ if mT_0 is used instead of T_0 (for $m \in \mathbf{N}$).
- The integral in the definition of $\langle \mathbf{f}, \mathbf{g} \rangle$ can be taken over any interval of length T_0 , since for any time offset T ,

$$\int_T^{T+T_0} f^*(t)g(t)dt = \int_0^{T_0} f^*(t)g(t)dt$$

This equality is based on the fact that $f^*(t)g(t)$ is also periodic with period T_0 . (**Your task:** show this equality either graphically, or by differentiating the integral on the left with respect to T .)

4. As it turns out, the Fourier sinusoids with frequencies $k\Omega_0$, where $k \in \mathbf{Z}$, are mutually orthogonal and have unit norm: if

$$v^{(k)}(t) = e^{jk\Omega_0 t} \quad \text{and} \quad v^{(\ell)}(t) = e^{j\ell\Omega_0 t},$$

then

$$\langle \mathbf{v}^{(k)}, \mathbf{v}^{(\ell)} \rangle = \begin{cases} 1, & k = \ell; \\ 0, & k \neq \ell. \end{cases}$$

Note that a similar relationship was obtained for discrete-time Fourier sinusoids, with the vector length N replacing 1.

5. To prove this relationship, we write

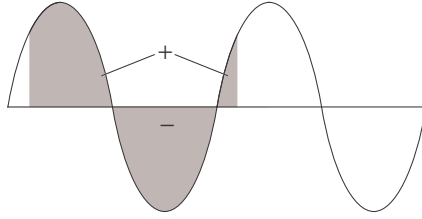
$$\langle \mathbf{v}^{(k)}, \mathbf{v}^{(\ell)} \rangle = \frac{1}{T_0} \int_0^{T_0} e^{-jk\Omega_0 t} e^{j\ell\Omega_0 t} dt = \frac{1}{T_0} \int_0^{T_0} e^{j(\ell-k)\Omega_0 t} dt$$

If $k = \ell$, then $e^{j(\ell-k)\Omega_0 t} = 1$ for all t , and thus

$$\langle \mathbf{v}^{(k)}, \mathbf{v}^{(\ell)} \rangle = \frac{1}{T_0} \int_0^{T_0} dt = 1$$

If $k \neq \ell$, then

$$\langle \mathbf{v}^{(k)}, \mathbf{v}^{(\ell)} \rangle = \frac{1}{T_0} \int_0^{T_0} \cos((\ell - k)\Omega_0 t) dt + \frac{j}{T_0} \int_0^{T_0} \sin((\ell - k)\Omega_0 t) dt,$$



where both sinusoids under the integral signs have common period $T_0/|\ell - k|$. The integral of any real sinusoid of nonzero frequency over an entire period equals zero (see figure above). Since the interval $[0, T_0)$ consists of exactly $|\ell - k|$ periods of $\cos((\ell - k)\Omega_0 t)$ and $\sin((\ell - k)\Omega_0 t)$, it follows that

$$\langle \mathbf{v}^{(k)}, \mathbf{v}^{(\ell)} \rangle = 0 + j \cdot 0 = 0$$

6. The synthesis equation (\spadesuit) is therefore a representation of $s(t)$ as a linear combination of *orthogonal* sinusoids. As was the case with vectors and the DFT, the coefficients $\{S_k\}$ can be obtained by projection onto each of the orthogonal components. Using boldface letters to indicate entire signals, we can rewrite (\spadesuit) as

$$\mathbf{s} = \sum_{\ell=-\infty}^{\infty} S_{\ell} \mathbf{v}^{(\ell)}$$

Taking inner products of both sides with $\mathbf{v}^{(k)}$, we obtain

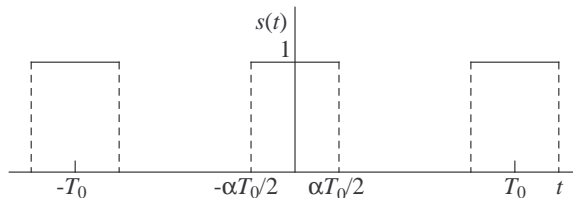
$$\begin{aligned} \langle \mathbf{v}^{(k)}, \mathbf{s} \rangle &= \sum_{\ell=-\infty}^{\infty} S_{\ell} \langle \mathbf{v}^{(k)}, \mathbf{v}^{(\ell)} \rangle \\ &= S_k \end{aligned}$$

The sought analysis equation is therefore $S_k = \langle \mathbf{v}^{(k)}, \mathbf{s} \rangle$, i.e.,

$$S_k = \frac{1}{T_0} \int_0^{T_0} s(t) e^{-jk\Omega_0 t} dt \quad (\clubsuit)$$

As noted before, the range of integration can be any interval of length T_0 .

7. **Example.** Let $s(t)$ be a rectangular pulse train with duty factor $\alpha \in (0, 1)$ and height 1, as shown below.



We first evaluate S_0 , also known as the *mean value* (or *DC offset*) of $s(t)$:

$$S_0 = \frac{1}{T_0} \int_0^{T_0} s(t) dt = \frac{\alpha T_0}{T_0} = \alpha$$

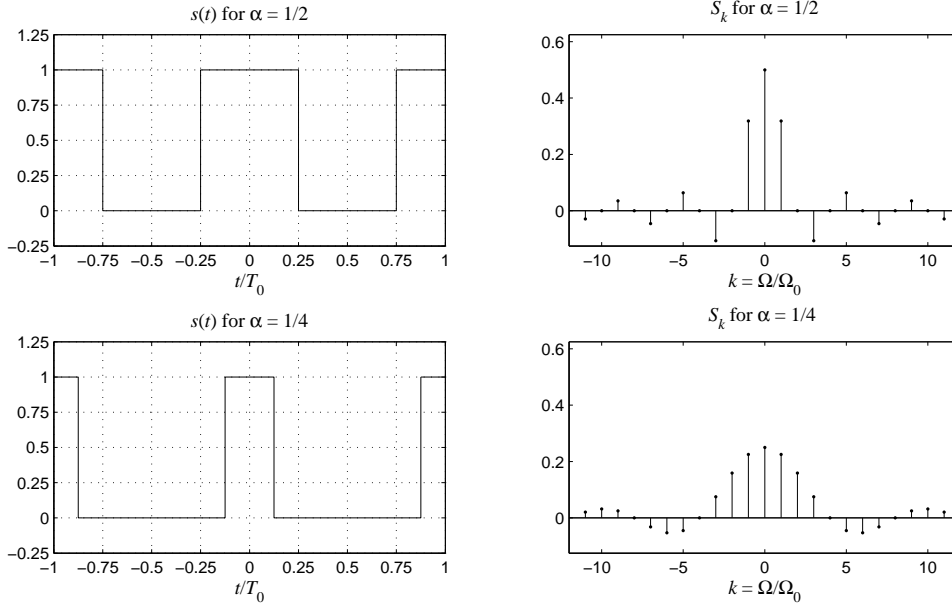
For $k \neq 0$, we have (using $(-T_0/2, T_0/2]$ instead of $[0, T_0)$ in (♣))

$$\begin{aligned} S_k &= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} s(t) e^{-jk\Omega_0 t} dt \\ &= \frac{1}{T_0} \int_{-\alpha T_0/2}^{\alpha T_0/2} \left\{ \cos(k\Omega_0 t) - j \sin(k\Omega_0 t) \right\} dt \\ &= \frac{1}{k\Omega_0 T_0} \left[\sin(k\Omega_0 t) + j \cos(k\Omega_0 t) \right]_{t=-\alpha T_0/2}^{\alpha T_0/2} \end{aligned}$$

Since $\sin(-\theta) = -\sin \theta$ and $\cos \theta = \cos(-\theta)$, we obtain

$$\begin{aligned} S_k &= \frac{2 \sin(k\alpha\Omega_0 T_0/2)}{k\Omega_0 T_0} \\ &= \frac{\sin(k\alpha\pi)}{k\pi} \end{aligned}$$

The graphs of $s(t)$ and S_k are shown below in two cases, $\alpha = 1/2$ and $\alpha = 1/4$.



8. A few remarks on the rectangular pulse train of the previous example:

- The signal is real-valued and symmetric (also known as *even*) about $t = 0$: $s(-t) = s(t)$. Accordingly, its Fourier series coefficients $\{S_k\}$ are also real-valued and symmetric:

$$S_{-k} = \frac{\sin(-k\alpha\pi)}{-k\pi} = \frac{\sin(k\alpha\pi)}{k\pi} = S_k$$

Combining the k^{th} and $(-k)^{\text{th}}$ complex sinusoids into a cosine, we obtain the alternative, cosine-only real series

$$s(t) = \alpha + 2 \sum_{k=1}^{\infty} \frac{\sin(k\alpha\pi)}{k\pi} \cdot \cos(k\Omega_0 t)$$

- The formula for S_k yields the correct answer for S_0 (namely α) in the limit as k (treated as a continuous variable) tends to 0. This is because $\sin \theta / \theta \rightarrow 1$ as $\theta \rightarrow 0$.

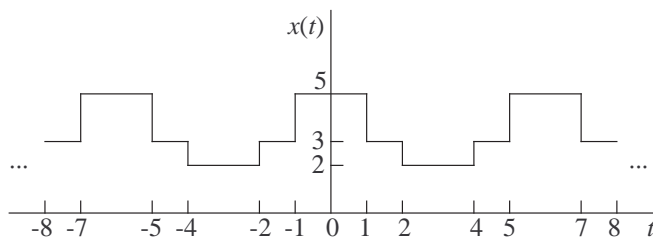
Questions:

- In the case $\alpha = 1/2$, S_k has a simple “non-trigonometric” form. What is it?
 - If $s(t)$ is scaled so that each pulse has area equal to T_0 (instead of αT_0), the modified coefficients S_k can be obtained by sampling a single function of θ which does not involve α . What is that function, and how is the sampling done?
9. The finite Fourier series approximation (\sum_{-K}^K , with K increasing) to the rectangular pulse train in the case $\alpha = 1/4$ can be illustrated graphically as follows:

```
To = 0.08; N = 500;
t = To*(-1 : 1/N : 1-1/N).';    % two periods shown
s = 0*t+1/4;
for K = 1:100
    s = s + 2*sin(K*pi/4)*cos(2*pi*K*t/To)/(K*pi);
    plot(t,s), title(['K = ' num2str(K)])
    pause
end
```

As was the case earlier with the sawtooth waveform, the finite sum approximation contains ripples which are prominent near the points of discontinuity. The same effect would be observed if $s(t)$ were processed by an *ideal lowpass filter* with cutoff frequency between $K f_0$ and $(K + 1) f_0$ (Hz).

10. **Example.** Consider the piecewise constant periodic waveform $x(t)$ (of period $T_0 = 6$) shown below.



By linearity, the Fourier series expansion of $x(t)$ has coefficients

$$X_0 = 2 + \frac{2}{3} + 2 \cdot \frac{1}{3} = \frac{10}{3}$$

and, for $k \neq 0$,

$$X_k = \frac{\sin(2k\pi/3)}{k\pi} + \frac{2 \sin(k\pi/3)}{k\pi}$$

Your task: Derive these formulas by expressing $x(t)$ as a sum of two rectangular pulse trains plus a constant.

LECTURE 20

Topics: further analogies between Fourier series and the DFT

Key Points:

- Circular operations (reversal and shifts) on a finite vector are equivalent to non-circular operations on the infinite periodic extension of that vector.

| <i>Time Domain Signal</i> | <i>Fourier Series Coefficients</i> |
|---|------------------------------------|
| $x(t)$ | X_k |
| $x^*(t)$ | X_{-k}^* |
| $x(-t)$ | X_{-k} |
| $x(t - T)$ | $e^{-jk\Omega_0 T} X_k$ |
| $(K \in \mathbf{Z}) \quad x(t)e^{jK\Omega_0 t}$ | X_{k-K} |
| $(\beta > 0) \quad x(\beta t)$ | X_k |

Theory and Examples:

1. The Fourier series and the discrete Fourier transform are parallel concepts. They both provide a decomposition of a periodic signal into mutually orthogonal, harmonically related sinusoids. The coefficients of these sinusoids collectively form the spectrum of the periodic signal.

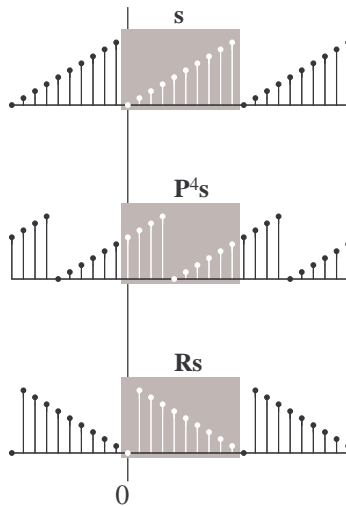
We have already seen many similarities between the two concepts, notably in the way the spectrum is obtained from the time-domain signal (i.e., by means of an inner product). The two concepts also exhibit similar properties in regard to signal transformations and time-frequency duality.

2. In this lecture we will explore some of these similarities, but also point out some important differences, between the Fourier series and the DFT. Most notably:
 - In the case of the DFT, the two domains—time and frequency—have a common structure, i.e., they are index sets of the same finite size. Periodicity is an optional feature: thus the DFT applies to both a finite-dimensional time-domain vector and its infinite periodic extension; while the set of Fourier frequencies has a counterpart in every interval of the form $[2r\pi, 2(r + 1)\pi)$, where $r \in \mathbf{Z}$.
 - In the case of the Fourier series, the two domains are very different. The time domain is continuous and inherently periodic. (Like the DFT, the Fourier series describes both a signal over a finite interval of time and its infinite periodic extension outside that interval.) On the other hand, the frequency domain is discrete, infinite (indexed by \mathbf{Z}) and distinctly non-periodic: we recall that, unlike discrete-time sinusoids, continuous-time sinusoids can have arbitrarily high frequencies. As a result of these differences, duality between time and frequency does not manifest itself as plainly as in the case of the DFT.

3. In the case of vectors, periodic extension of the time domain makes it possible to implement certain circular time index permutations defined on $\{0, \dots, L - 1\}$ in terms of equivalent non-circular operations performed on \mathbf{Z} .

Specifically, we consider the circular right m -shift \mathbf{P}^m and the circular reversal \mathbf{R} performed on a signal vector \mathbf{s} of size L . As the figure below illustrates, these permutations can be carried out by

- periodically extending \mathbf{s} from indices $\{0, \dots, L - 1\}$ to all indices in \mathbf{Z} ;
- performing either a regular (i.e., non-circular) right shift or a regular reversal on the periodic sequence; and
- restricting the resulting sequence to indices $\{0, \dots, L - 1\}$.



4. In the case of a continuous-time signal $s(t)$, the time shift $t \rightarrow t - T$ (where T is fixed) and the time reversal $t \rightarrow -t$ are both regular (non-circular) operations. If $s(t)$ is periodic with period T_0 , these operations result in a circular shift and a circular reversal (respectively) of the first period $\{s(t), 0 \leq t < T_0\}$.
5. The Fourier series properties given below are analogous to familiar DFT properties. A good intuition about the DFT generally suffices to state these properties correctly (i.e., as educated guesses). Formal proofs are also straightforward using either the synthesis (sum) or analysis (integral) equations:

$$s(t) = \sum_{k=-\infty}^{\infty} S_k e^{jk\Omega_0 t} \quad \Leftrightarrow \quad S_k = \frac{1}{T_0} \int_0^{T_0} s(t) e^{-jk\Omega_0 t} dt$$

6. The first group of properties is analogous to **DFT 2-4**, and pertains to complex conjugates and index reversals:

$$y(t) = x^*(t) \quad \Rightarrow \quad Y_k = X_{-k}^* \quad (1)$$

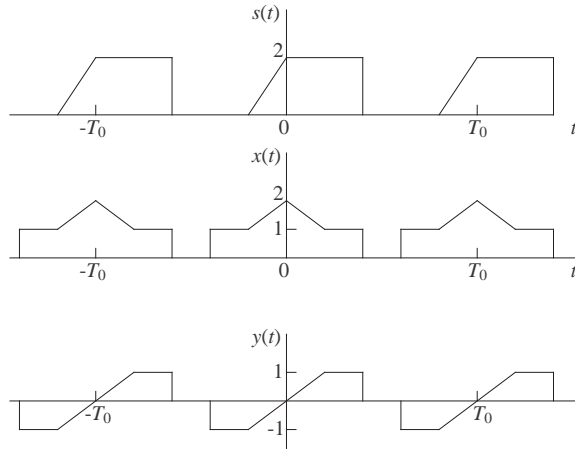
$$x(t) = x^*(t) \quad \Rightarrow \quad X_k = X_{-k}^* \quad (2)$$

$$y(t) = x(-t) \quad \Rightarrow \quad Y_k = X_{-k} \quad (3)$$

Note that both time and frequency reversals are non-circular (i.e., $t \rightarrow -t$ and $k \rightarrow -k$).

Property (2), which follows directly from (1), is the familiar fact that a real-valued signal has a conjugate symmetric spectrum.

7. **Example.** Consider the three signals $s(t)$, $x(t)$ and $y(t)$ shown below. We would like to express the Fourier series coefficients of $x(t)$ and $y(t)$ in terms of those of $s(t)$ (assumed known).



First, we note that

$$(\forall t) \quad x(t) = \frac{s(t) + s(-t)}{2} \quad \text{and} \quad y(t) = \frac{s(t) - s(-t)}{2}$$

Also,

$$(\forall t) \quad x(t) = x(-t) \quad \text{and} \quad y(t) = -y(-t)$$

The signals $x(t)$ and $y(t)$ are respectively known as the *even* and *odd parts* of $s(t)$, and their sum equals $s(t)$.

From (1), it follows that

$$(\forall k) \quad X_k = \frac{S_k + S_{-k}}{2} \quad \text{and} \quad Y_k = \frac{S_k - S_{-k}}{2}$$

These expressions can be further simplified. Since $s(t)$ is real-valued, it follows from (2) that $S_{-k} = S_k^*$, and thus

$$(\forall k) \quad X_k = \Re\{S_k\} = \Re\{S_{-k}\} = X_{-k}$$

and

$$Y_k = j\Im\{S_k\} = -j\Im\{S_{-k}\} = -Y_{-k}$$

Note that X_k is real and even (i.e., symmetric) about $k = 0$. This is to be expected since $x(t)$ is also real and even about $t = 0$.

8. The counterpart of **DFT 5** (i.e., $\mathbf{P}^m \mathbf{s} \longleftrightarrow \mathbf{F}^{-m} \mathbf{S}$) is the *time shift*, or *time delay*, property

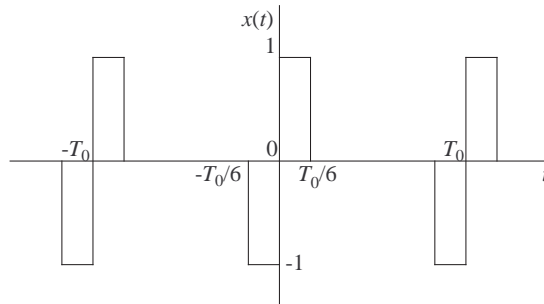
$$y(t) = x(t - T) \quad \Rightarrow \quad Y_k = e^{-jk\Omega_0 T} X_k \quad (4)$$

In the frequency domain, the spectrum is multiplied by a complex sinusoid in k , i.e., its phase is shifted by a linear function of k . The same effect was seen in **DFT 5**.

9. **Example.** The Fourier series coefficients of rectangular pulse train were derived in the previous lecture. Using that result and (4), we obtain, for the signal below,

$$\begin{aligned} X_k &= (e^{-jk\pi/6} - e^{jk\pi/6}) \cdot \frac{\sin(k\pi/6)}{k\pi} \\ &= -j \cdot \frac{2}{k\pi} \sin^2\left(\frac{k\pi}{6}\right) \end{aligned}$$

As expected, $X_{-k} = X_k^*$. The fact that X_k is purely imaginary (and thus also equal to $-X_{-k}$) is due to the odd symmetry in $x(t)$, i.e., $x(-t) = -x(t)$.



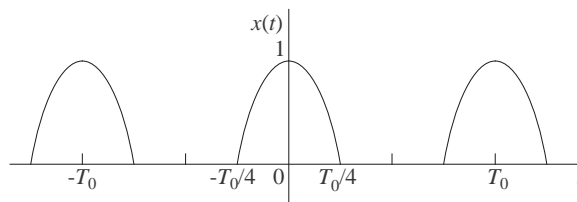
Your Task: Show the missing details in the derivation of X_k .

10. The counterpart of **DFT 6** (i.e., $\mathbf{F}^m \mathbf{s} \longleftrightarrow \mathbf{P}^m \mathbf{S}$) is the *modulation*, or *frequency shift*, property

$$y(t) = x(t)e^{jK\Omega_0 t} \quad \Rightarrow \quad Y_k = X_{k-K} \quad (5)$$

The time-domain signal is multiplied by the K^{th} Fourier sinusoid. Correspondingly, the spectrum undergoes a right shift by K indices.

11. **Example.** A half-wave rectified sinusoid (such as the one shown below) can be obtained by multiplying a full-wave sinusoid by a half-duty ($\alpha = 1/2$) rectangular pulse train. **Your task:** Using dotted lines, sketch the full-wave sinusoid and the half-duty pulse train on top of $x(t)$.



If $s(t)$ is the symmetric rectangular half-duty pulse train of unit height and $\Omega_0 = 2\pi/T_0$, then

$$x(t) = s(t) \cos(\Omega_0 t) = s(t) \cdot \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$$

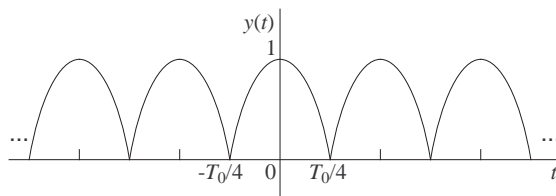
Therefore

$$\begin{aligned} X_k &= \frac{S_{k-1} + S_{k+1}}{2} \\ &= \frac{\sin((k-1)\pi/2)}{2(k-1)\pi} + \frac{\sin((k+1)\pi/2)}{2(k+1)\pi} \end{aligned}$$

Noting that all nonzero $\sin(\cdot)$ values above equal ± 1 , and recalling that the value of $\sin \theta/\theta$ can be taken as 1 at $\theta = 0$, we obtain the simplified form

$$X_k = \begin{cases} 1/4, & k = -1, 1 \\ \frac{(-1)^{k/2}}{\pi(1-k^2)}, & k \text{ even (including 0)} \\ 0, & \text{all other } k \end{cases}$$

Your task: Determine the Fourier series coefficients Y_k of the full-wave rectified sinusoid $y(t)$ below in terms of the X_k 's. The period of $y(t)$ is half of that of $x(t)$, and accordingly, the Y_k 's are expressed in terms of X_{2k} 's.



12. Point-by-point multiplication of two signals has the same meaning, regardless of whether the signals are vectors, sequences or continuous-time waveforms. In the context of the DFT (i.e., where vectors are involved), the dual property is circular convolution. This duality is also seen in the Fourier series provided convolution is properly defined.

- *Convolution of two periodic signals $x(t)$ and $y(t)$ of common period T_0 .* As in the case of the DFT, this is defined in a circular (or periodic) fashion using an unconjugated dot product between either signal and a reversed/shifted version of the other. The dot product is an integral, and the resulting continuous-time signal is

$$(\forall t) \quad s(t) = \frac{1}{T_0} \int_0^{T_0} x(\tau) y(t - \tau) d\tau$$

As with **DFT 9**, convolution in the time domain is equivalent to point-by-point multiplication in the frequency domain:

$$(\forall k) \quad S_k = X_k Y_k$$

- *Convolution of two infinite sequences* $\{X_k, k \in \mathbf{Z}\}$ and $\{Y_k, k \in \mathbf{Z}\}$. The result is the sequence $\{S_k, k \in \mathbf{Z}\}$ given by the infinite sum

$$S_k = \sum_{\ell=-\infty}^{\infty} X_{\ell} Y_{k-\ell}$$

(Note that this operation cannot be circular or periodic, since \mathbf{Z} has no inherent periodicity.) As with **DFT 8**, the time-domain equivalent is point-by-point multiplication:

$$(\forall t) \quad s(t) = x(t)y(t)$$

- 13.** You may have noted in the examples given so far that the final expression for the Fourier series coefficients does not contain the period T_0 . This is a general fact: time dilation or compression (i.e., stretching or shrinking a signal horizontally) by means of the transformation $t \rightarrow \beta t$ (where $\beta > 0$) has no effect on the Fourier series coefficients. Indeed,

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t} \quad \Rightarrow \quad x(\beta t) = \sum_{k=-\infty}^{\infty} X_k e^{jk(\beta\Omega_0)t}$$

and therefore

$$y(t) = x(\beta t) \quad \Rightarrow \quad Y_k = X_k \tag{6}$$

This is known as the *time scaling* property of the Fourier series, and has no direct counterpart in the DFT.

LECTURE 21

Topics: introduction to linear time-invariant filters; response to FIR filters to sinusoidal and exponential inputs; frequency response and system function

Textbook References: sections 4.2.2, 4.3, 4.4.1, 4.4.3

Key Points:

- The input-output relationship

$$y[n] = b_0x[n] + b_1x[n-1] + \cdots + b_Mx[n-M], \quad n \in \mathbf{Z},$$

describes a linear time-invariant system known as a finite impulse response (FIR) filter.

- If the input $x[\cdot]$ to a FIR filter is the complex exponential sequence $x[n] = z^n$ (where $z \in \mathbf{C}$), the output $y[\cdot]$ is given by $y[n] = H(z)z^n$, where

$$H(z) = b_0 + b_1z^{-1} + \cdots + b_Mz^{-M}$$

is the filter's system function. In particular, when $x[n] = e^{j\omega n}$, the output is given $y[n] = H(e^{j\omega})e^{j\omega n}$, where

$$H(e^{j\omega}) = b_0 + b_1e^{-j\omega} + \cdots + b_Me^{-j\omega M}$$

is the filter's frequency response.

- If all coefficients b_k are real-valued, then the amplitude response $|H(e^{j\omega})|$ is symmetric (even) about $\omega = 0$ and π , while the phase response $\angle H(e^{j\omega})$ is antisymmetric (odd) about the same frequencies.
- The frequency response of an FIR filter with coefficient vector $\mathbf{b} = b_{0:M}$ can be obtained for $N \geq M + 1$ uniformly spaced frequencies in $[0, 2\pi)$ by zero-padding \mathbf{b} to length N and computing a DFT.

Theory and Examples:

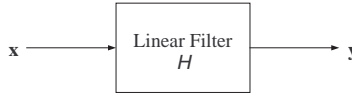
1. In developing the DFT, we considered discrete-time signals which are either vectors (i.e., consisted of finitely many samples) or periodic extensions thereof. We now turn our attention to general discrete-time signals, namely sequences such as

$$\mathbf{x} = x[\cdot] = \{x[n], n \in \mathbf{Z}\}$$

If \mathbf{x} is a linear combination of (not necessarily periodic) sinusoids, then it also has a *spectrum*. Its spectrum consists of the coefficients of these sinusoids given (or plotted) as a function of frequency.

2. Linear filters can be used to alter the spectra of sequences in an immediate, “real time”, fashion. A linear filter \mathcal{H} acts as a (linear) transformation of an input sequence \mathbf{x} to an output sequence \mathbf{y} . (Since the input and output spaces are infinite-dimensional, this linear transformation is not represented by a finite-dimensional matrix.)

$$\mathbf{y} = \mathcal{H}(\mathbf{x})$$



3. Analytically, the simplest discrete-time filter is the so-called *finite impulse response* (FIR) filter. At each sampling instant n , a new input sample is read in and stored in a buffer containing the $M + 1$ most recent input samples, i.e., $x[n - M : n]$. These samples are linearly combined using a fixed vector \mathbf{b} of coefficients to produce an output sample $y[n]$. This procedure is described by a single formula:

$$y[n] = b_0x[n] + b_1x[n - 1] + \cdots + b_Mx[n - M], \quad n \in \mathbf{Z},$$

known as the filter *input-output relationship*.

4. An FIR filter has two notable properties:

- *Linearity*. If three *identical* filters are used on input sequences $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$, then the output sequence of the third filter is the same (in terms of coefficients) linear combination of the output sequences of the first two filters, i.e., $\mathbf{y}^{(3)} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)}$.
- *Time invariance*. If two identical filters are used on two input sequences which are time-delayed versions of each other, then the observed output sequences will also be time-delayed versions of each other (with the same delay).

Your Task: Prove these properties formally.

5. An FIR filter modifies the spectrum of an input sequence \mathbf{x} , i.e, it changes the amounts (coefficients) of the sinusoidal components of \mathbf{x} . To illustrate this point, we take a single complex sinusoid of frequency ω :

$$x[n] = e^{j\omega n}, \quad n \in \mathbf{Z}$$

and put it through a filter with coefficient vector $b = [1 \ 2 \ 2 \ 1]^T$. The output sequence \mathbf{y} is given by

$$\begin{aligned} y[n] &= x[n] + 2x[n - 1] + 2x[n - 2] + x[n - 3] \\ &= e^{j\omega n} + 2e^{j\omega(n-1)} + 2e^{j\omega(n-2)} + e^{j\omega(n-3)} \\ &= (1 + 2e^{-j\omega} + 2e^{-j2\omega} + e^{-j3\omega}) \cdot e^{j\omega n} \\ &= (1 + 2e^{-j\omega} + 2e^{-j2\omega} + e^{-j3\omega}) \cdot x[n] \end{aligned}$$

Thus the output is a complex sinusoid of the *same* frequency; the filter merely scales the input by a complex factor which depends on the frequency ω . This is true for any FIR filter.

6. The scaling factor above is known as the *frequency response* of the filter and is denoted by $H(e^{j\omega})$:

$$H(e^{j\omega}) = 1 + 2e^{-j\omega} + 2e^{-j2\omega} + e^{-j3\omega}$$

It is a polynomial in (negative) powers of $e^{j\omega}$. The expression for $H(e^{j\omega})$ can be simplified by noting the (non-circular) symmetry of the coefficient vector

$$\mathbf{b} = b_{0:3} = [1 \ 2 \ 2 \ 1]^T$$

about the “middle” index $3/2$. Factoring out $e^{-j3\omega/2}$, we obtain

$$\begin{aligned} H(e^{j\omega}) &= e^{-j3\omega/2} \cdot (e^{j3\omega/2} + 2e^{j\omega/2} + 2e^{-j\omega/2} + e^{-j3\omega/2}) \\ &= e^{-j3\omega/2} \cdot (4 \cos(\omega/2) + 2 \cos(3\omega/2)) \end{aligned}$$

7. The modulus $|H(e^{j\omega})|$ of the frequency response is known as the *amplitude*, or *magnitude*, response of the filter. In this case, noting that $|e^{j\theta}| = 1$, we have

$$|H(e^{j\omega})| = |4 \cos(\omega/2) + 2 \cos(3\omega/2)|$$

This function of ω is symmetric about $\omega = \pi$ and has three zeros in the interval $[0, 2\pi)$: at $\omega = 2\pi/3$, π and $4\pi/3$. Thus for any of the three input sequences

$$x^{(1)}[n] = e^{j2\pi n/3}, \quad x^{(2)}[n] = (-1)^n \quad \text{and} \quad x^{(3)}[n] = e^{j4\pi n/3}, \quad n \in \mathbf{Z},$$

the filter output equals 0 for all n . By linearity, the same is true about any linear combination of these three input sequences; in particular, the sequence $x[n] = \cos(2\pi n/3 + \phi_0)$. (**Question:** Why?)

8. The angle $\angle H(e^{j\omega})$ of the frequency response is known as the phase response of the filter. In this case, recalling that $\angle z_1 z_2 = \angle z_1 + \angle z_2$, we have

$$\angle H(e^{j\omega}) = -\frac{3\omega}{2} + \angle (4 \cos(\omega/2) + 2 \cos(3\omega/2))$$

The first term is linear in ω , while the second equals 0 or $\pm\pi$, depending on whether the real number $4 \cos(\omega/2) + 2 \cos(3\omega/2)$ is positive or negative. This means that $\angle H(e^{j\omega})$ is piecewise linear with constant slope ($= -3/2$) and discontinuities of size π occurring wherever $4 \cos(\omega/2) + 2 \cos(3\omega/2)$ changes sign, i.e., at $\omega = 2\pi/3$, $\omega = \pi$ and $\omega = 4\pi/3$.

Note: The phase response $\angle H(e^{j\omega})$ is piecewise linear for all FIR filters whose coefficients have even or odd symmetry about the middle index ($M/2$).

9. The expression for the frequency response $H(e^{j\omega})$ is always periodic with period (in ω) equal to 2π ; as a result, $H(e^{j\omega})$ is (automatically) periodically extended outside the interval $[0, 2\pi)$.

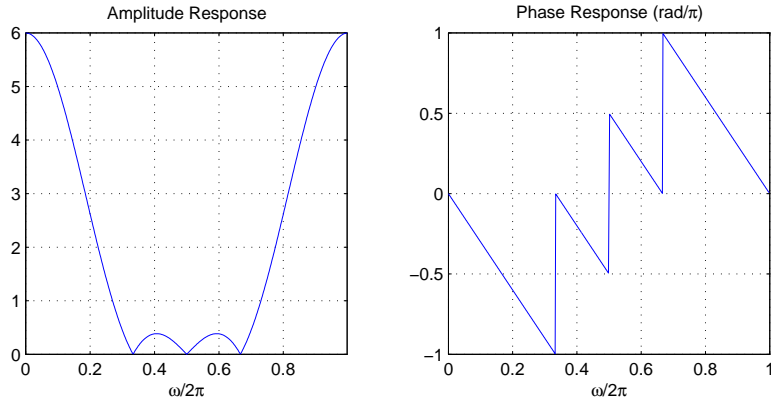
The amplitude response $|H(e^{j\omega})|$ is symmetric (even) about $\omega = 0$ and π , while the phase response $\angle H(e^{j\omega})$ is antisymmetric (odd) about the same frequencies. This is true for all FIR filters with *real-valued* coefficients b_0, \dots, b_M , since

$$H(e^{j\omega}) = b_0 + b_1 e^{-j\omega} + \dots + b_M e^{-j\omega M}$$

and

$$H(e^{-j\omega}) = H(e^{j(2\pi-\omega)}) = b_0 + b_1 e^{j\omega} + \dots + b_M e^{j\omega M} = H^*(e^{j\omega})$$

(Recall that $|z^*| = |z|$ and $\angle z^* = -\angle z$.)



10. Plotting $|H(e^{j\omega})|$ and $\angle H(e^{j\omega})$ requires computing $H(e^{j\omega})$ at a sufficiently dense set of frequencies ω in $[0, 2\pi)$, i.e.,

$$\omega = 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \dots, 1 - \frac{2\pi}{N}, \quad \text{where } N \gg M$$

These are the Fourier frequencies for a vector of length N . Provided $N \geq M + 1$, the resulting vector of values of $H(e^{j\omega})$ equals the DFT of

$$[\mathbf{b}; \mathbf{0}_{N-M-1}]$$

11. **Example.** The following MATLAB script computes the amplitude and phase responses of the FIR filter with input-output relationship

$$y[n] = x[n] + 2x[n-1] + 2x[n-2] + x[n-3]$$

at $N = 512$ uniformly spaced frequencies in $[0, 2\pi)$.

```

b = [1 2 2 1].';
H = fft(b,512);
A = abs(H);
q = angle(H);
w = (0: 1/512 : 1-1/512).*(2*pi);
plot(w,A);
plot(w,q);

```

12. Similar conclusions can be drawn when the input signal is a complex exponential, i.e.,

$$x[n] = z^n, \quad n \in \mathbf{Z}$$

where z is an arbitrary complex number. Using the same filter as previously, we have, for all n ,

$$\begin{aligned}
y[n] &= z^n + 2z^{n-1} + 2z^{n-2} + z^{n-3} \\
&= (1 + 2z^{-1} + 2z^{-2} + z^{-3}) \cdot z^n \\
&= (1 + 2z^{-1} + 2z^{-2} + z^{-3}) \cdot x[n]
\end{aligned}$$

The complex scaling factor

$$H(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$$

is known as the *system function* of the filter. The frequency response $H(e^{j\omega})$ of an FIR filter is the restriction of its system function to the unit circle $z = e^{j\omega}$.

- 13. Example.** Verify the following input-pairs for the FIR filter considered above (all relationships hold for all n).

$$\begin{aligned}x[n] = 3^n &\Rightarrow y[n] = \frac{52}{27} \cdot 3^n \\x[n] = (-3)^n &\Rightarrow y[n] = \frac{14}{27} \cdot (-3)^n \\x[n] = \left(\frac{1}{3}\right)^n &\Rightarrow y[n] = 52 \left(\frac{1}{3}\right)^n\end{aligned}$$

LECTURE 22

Topics: response of FIR filters to exponential inputs (continued); response of FIR filters to periodic inputs; cascaded filters

Textbook References: sections 4.4.2, 4.4.4, 4.4.5

Key Points:

- For an FIR filter with frequency response $H(e^{j\omega})$,

$$x[n] = r^n \cos(\omega_0 n + \phi) \quad \Rightarrow \quad y[n] = |H(re^{j\omega_0})| r^n \cos(\omega_0 n + \phi + \angle H(re^{j\omega_0}))$$

- The response of an FIR filter to a periodic input can be computed by a circular convolution in the time domain, or, equivalently, by taking an element-wise product of DFT's in the frequency domain. Either technique can be used to generate an integer number of periods of the output signal.
- If two filters with system functions $H_1(z)$ and $H_2(z)$ are connected in series (cascade), the resulting filter has system function

$$H(z) = H_1(z)H_2(z) ,$$

regardless of the order of the connection.

Theory and Examples:

1. We have seen that if $x[n] = z^n$ is the input to an FIR filter with coefficients b_0, \dots, b_M , then the output $y[\cdot]$ is given by $y[n] = H(z)z^n$, where

$$H(z) = b_0 + b_1 z^{-1} + \dots + b_M z^{-M}$$

is the filter's system function.

2. If the input is the real-valued sinusoid

$$x[n] = \cos(\omega_0 n + \phi) = \frac{1}{2} \cdot e^{j\phi} e^{j\omega_0 n} + \frac{1}{2} \cdot e^{-j\phi} e^{-j\omega_0 n} ,$$

then, by linearity, the output is given by

$$y[n] = \frac{1}{2} \cdot H(e^{j\omega_0}) e^{j\phi} e^{j\omega_0 n} + \frac{1}{2} \cdot H(e^{-j\omega_0}) e^{-j\phi} e^{-j\omega_0 n}$$

Since $H(e^{-j\omega}) = H^*(e^{j\omega})$, the expression above equals the sum of two complex conjugate terms, which is the same as twice the real part of either term:

$$y[n] = \Re \left\{ H(e^{j\omega_0}) e^{j(\omega_0 n + \phi)} \right\}$$

Writing $H(e^{j\omega_0})$ in complex exponential form, i.e., $H(e^{j\omega}) = |H(e^{j\omega})| e^{j\angle H(e^{j\omega})}$, we obtain

$$y[n] = |H(e^{j\omega_0})| \cos(\omega_0 n + \phi + \angle H(e^{j\omega_0})) \quad (n \in \mathbf{Z})$$

The same approach can be applied to the *oscillating* exponential input

$$x[n] = r^n \cos(\omega_0 n + \phi) = \frac{e^{j\phi}}{2} \cdot r^n e^{j\omega_0 n} + \frac{e^{-j\phi}}{2} \cdot r^n e^{-j\omega_0 n}$$

Taking $z = re^{\pm j\omega_0}$, we obtain in this case

$$y[n] = |H(re^{j\omega_0})| r^n \cos(\omega_0 n + \phi + \angle H(re^{j\omega_0})) \quad (n \in \mathbf{Z})$$

3. Example. Let

$$x[n] = 2^{-n} \cdot \cos\left(\frac{\pi n}{3} + \frac{\pi}{4}\right), \quad n \in \mathbf{Z}$$

and (as before)

$$y[n] = x[n] + 2x[n-1] + 2x[n-2] + x[n-3]$$

Setting $z = (e^{j\pi/3})/2$, we obtain

$$H(z) = 1 + 4e^{-j(\pi/3)} + 8e^{-j(2\pi/3)} + 8e^{-j\pi} = 13.748 \cdot e^{-j2.285}$$

The output sequence is therefore given by

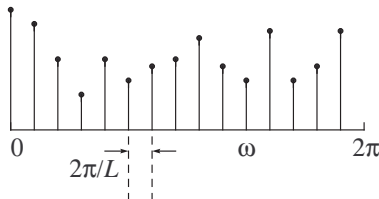
$$y[n] = 13.748 \cdot 2^{-n} \cdot \cos\left(\frac{\pi n}{3} - 1.499\right), \quad n \in \mathbf{Z}$$

Your task: Repeat for $x[n] = \cos(\pi n/3 + \pi/4)$.

4. Periodic sequences are always expressible as sums of sinusoids. We have seen that if $x[\cdot]$ is periodic with period L , then it can be written as

$$x[n] = \frac{1}{L} \sum_{k=0}^{L-1} \mathcal{X}[k] e^{jk(2\pi/L)n}, \quad n \in \mathbf{Z}$$

where $\mathcal{X}[0 : L-1]$ is the DFT of its *first period* $x[0 : L-1]$. Thus $x[\cdot]$ is a linear combination of L (or fewer) complex sinusoids, whose frequencies are multiples of $2\pi/L$. Qualitatively, a graph of its (magnitude) spectrum would look like this:



5. Example. Suppose

$$x[n] = A_1 \cos(2\pi f_1 n + \phi_1) + A_2 \cos(2\pi f_2 n + \phi_2) + A_3 \cos(2\pi f_3 n + \phi_3),$$

where the A_i 's are real and nonzero, and

$$f_1 = \frac{1}{8}, \quad f_2 = \frac{3}{20} \quad \text{and} \quad f_3 = \frac{5}{12}$$

Each f_i is rational, therefore each sinusoid is periodic. Their sum $x[\cdot]$ is also periodic, and its period is the smallest value of L for which all three frequencies are multiples of $1/L$. Thus L equals the least common multiple of 8, 20 and 12, namely $L = 120$. Obviously, the spectrum of $x[\cdot]$ has only six (out of 120 possible) lines in $[0, 2\pi)$.

Note: In discrete time, the sum of two or more periodic signals is always periodic. This is not true in continuous time.

6. If the periodic signal $x[\cdot]$ from above is the input to an FIR filter with frequency response $H(e^{j\omega})$, then, by linearity, the filter output is given by

$$y[n] = \frac{1}{L} \sum_{k=0}^{L-1} H(e^{jk(2\pi/L)}) \mathcal{X}[k] e^{jk(2\pi/L)n}, \quad n \in \mathbf{Z}$$

Thus the output sequence $y[\cdot]$ is also periodic with period L , and its first period $y[0 : L - 1]$ has DFT $\mathcal{Y}[0 : L - 1]$ given by

$$\mathcal{Y}[k] = H(e^{jk(2\pi/L)}) \mathcal{X}[k], \quad k = 0 : L - 1 \quad (\spadesuit)$$

7. As we saw earlier, $H(e^{j\omega}) = \sum_{n=0}^M b_n e^{-j\omega n}$ can be computed for any set of $M + 1$ or more uniformly spaced frequencies by zero-padding the vector \mathbf{b} and computing a DFT. Thus (\spadesuit) suggests a way of computing the response of an FIR filter to a periodic input of period L (where $L \geq M + 1$) using a frequency domain-based tool, namely the element-wise multiplication of two DFT's.

8. **Example.** Consider the filter with input-output relationship

$$y[n] = x[n] - 4x[n - 1] + x[n - 2]$$

Suppose that the input $x[\cdot]$ is periodic with period $L = 4$, such that

$$x[0 : 3] = [2 \quad 1 \quad -3 \quad 5]^T$$

The MATLAB script below computes the first period $y[0 : 3]$ of the output.

```
x = [2 1 -3 5].';
X = fft(x);
b = [1 -4 1].';
H = fft(b,4);
Y = H.*X;
y = ifft(Y);
```

9. Since element-wise multiplication of DFT's is equivalent to circular convolution in the time domain, (\spadesuit) suggests that the response of an FIR filter to a periodic input can be computed using a circular convolution in the time domain. *This is not surprising:* by rewriting the input-equation $y[n] = x[n] - 4x[n - 1] + x[n - 2]$ in the previous example as

$$y[n] = x[n] - 4x[n - 1] + x[n - 2] + 0 \cdot x[n - 3],$$

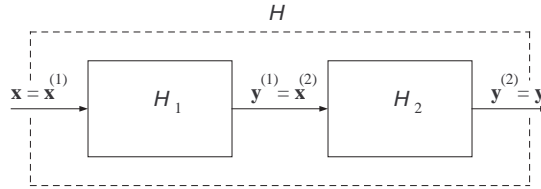
we obtain

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \begin{bmatrix} 2 & 5 & -3 & 1 \\ 1 & 2 & 5 & -3 \\ -3 & 1 & 2 & 5 \\ 5 & -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ 1 \\ 0 \end{bmatrix}$$

This is the circular convolution of $x[0 : 3]$ and $[\mathbf{b}; 0]$, which is *precisely* what the MATLAB script (above) computes.

10. **Question:** How would the computations in the last two items change if the same input sequence $x[\cdot]$ were processed instead by an FIR filter with $M + 1 = 7$ coefficients?
11. If two FIR filters are connected in series, or in a cascade (as shown below), the resulting system function is given by the product of the two system functions, i.e.,

$$H(z) = H_1(z)H_2(z)$$



(Note that the order in which the two filters are connected is immaterial.) This is proved by using $x[n] = z^n$ as the input to the cascade. The output of the first filter is

$$y^{(1)}[n] = H_1(z)z^n, \quad (n \in \mathbf{Z})$$

and, by linearity, the output of the second filter (same as the output of the cascade) is

$$y^{(2)}[n] = y[n] = H_1(z)H_2(z)z^n = H_2(z)H_1(z)z^n \quad (n \in \mathbf{Z})$$

12. **Example.** Two FIR filters with coefficient vectors

$$\begin{aligned} \mathbf{b}^{(1)} &= [1 \quad 2 \quad 2 \quad 1]^T \\ \mathbf{b}^{(2)} &= [1 \quad -4 \quad 1]^T \end{aligned}$$

are connected in cascade. The resulting filter has system function

$$\begin{aligned} H(z) &= (1 + 2z^{-1} + 2z^{-2} + z^{-3})(1 - 4z^{-1} + z^{-2}) \\ &= 1 - 2z^{-1} - 5z^{-2} - 5z^{-3} - 2z^{-4} + z^{-5} \end{aligned}$$

and is therefore an FIR filter with coefficient vector

$$\mathbf{b} = [1 \quad -2 \quad -5 \quad -5 \quad -2 \quad 1]^T$$

and input-output relationship

$$y[n] = x[n] - 2x[n-1] - 5x[n-2] - 5x[n-3] - 2x[n-4] + x[n-5]$$

LECTURE 23

Topics: linear convolution of sequences and vectors; linear convolution as circular convolution; block convolution

Textbook References: sections 4.6, 4.7.1, 4.7.2 and 4.7.4

Key Points:

- The input $\mathbf{x} = x[\cdot]$ and output $\mathbf{y} = y[\cdot]$ of a linear time-invariant system are related by the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \stackrel{\text{def}}{=} (\mathbf{h} * \mathbf{x})[n]$$

The sequence $\mathbf{h} = h[\cdot]$ is the system's response to a unit impulse $\boldsymbol{\delta} = \delta[\cdot]$.

- If \mathbf{h} and \mathbf{x} are finite-duration sequences with activity intervals $0 : K - 1$ and $0 : L - 1$ respectively, then $\mathbf{y} = \mathbf{h} * \mathbf{x}$ is also a finite-duration sequence with activity interval $0 : K + L - 2$. This serves as an implicit definition for the linear convolution of two vectors:

$$y[0 : K + L - 2] = h[0 : K - 1] * x[0 : L - 1]$$

- If $\mathbf{b} = b[0 : K - 1]$ and $\mathbf{s} = s[0 : L - 1]$, then

$$\mathbf{b} * \mathbf{s} = [\mathbf{b}; \mathbf{0}_{L-1}] \circledast [\mathbf{s}; \mathbf{0}_{K-1}]$$

Thus linear convolution of vectors can be implemented by means of circular convolution—and thus optionally, DFT's.

- If $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ have lengths L_1 and L_2 , respectively, then

$$\mathbf{b} * [\mathbf{s}^{(1)}; \mathbf{s}^{(2)}] = [\mathbf{b} * \mathbf{s}^{(1)}; \mathbf{0}_{L_2}] + [\mathbf{0}_{L_1}; \mathbf{b} * \mathbf{s}^{(2)}]$$

Theory and Examples:

1. In the preceding lectures, we examined in detail the response of an FIR filter to two types of inputs: infinite-duration (two-sided) exponentials and periodic sequences. We will now broaden our scope to arbitrary input sequences $x[\cdot]$, and discuss the implications of the input-output relationship

$$y[n] = \sum_{k=0}^M b_k x[n-k], \quad n \in \mathbf{Z}$$

We note that the same value for $y[n]$ is obtained if we pad \mathbf{b} with infinitely many zero coefficients on both sides, i.e., using the sequence $h[\cdot]$ defined

$$h[n] = \begin{cases} b_n, & 0 \leq n \leq M; \\ 0, & \text{otherwise,} \end{cases}$$

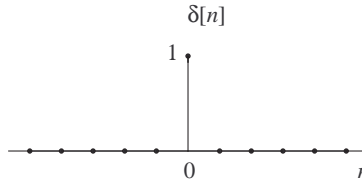
instead of \mathbf{b} . The input-output relationship above becomes

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k], \quad n \in \mathbf{Z} \quad (\spadesuit)$$

2. The sequence $h[\cdot]$ is known as the *impulse response* of the FIR filter with coefficient vector **b**. The unit impulse sequence $\delta[\cdot]$ is defined by

$$\delta[n] = \begin{cases} 1, & n = 0; \\ 0, & \text{otherwise,} \end{cases}$$

and is shown below.



It is easy to show that

$$x[\cdot] = \delta[\cdot] \quad \Rightarrow \quad y[\cdot] = h[\cdot],$$

i.e., $h[\cdot]$ is the response of the filter to a unit impulse. (**Your task:** Convince yourself that this is so.)

3. The sum in (\spadesuit) is known as the (non-circular) *convolution* of sequences $\mathbf{h} = h[\cdot]$ and $\mathbf{x} = x[\cdot]$. When computed for all n , it defines a new sequence $\mathbf{y} = y[\cdot]$. Symbolically, we write

$$\mathbf{y} = \mathbf{h} * \mathbf{x}$$

Convolution is commutative in its two arguments. This can be shown by a change in the summation variable, i.e., $k' = n - k$:

$$\sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k'=-\infty}^{\infty} h[n-k']x[k']$$

Thus

$$\mathbf{x} * \mathbf{h} = \mathbf{h} * \mathbf{x}$$

4. FIR filters are characterized by the property that $h[\cdot]$ has finite duration, i.e., it takes only a finite number of nonzero values. Other linear time-invariant systems encountered or used in practice have impulse responses of infinite duration. The convolution sum (\spadesuit) is valid for all such systems, i.e., it is a *universal input-output relationship* in terms of a single characteristic, namely the response of the system to a unit impulse. To see why $h[\cdot]$ can play that role, note that by time invariance,

$$x[\cdot - k] = \delta[\cdot - k] \quad \Rightarrow \quad y[\cdot - k] = h[\cdot - k]$$

for any k . Any input $x[\cdot]$ can be expressed as a linear combination of time-shifted impulses, i.e.,

$$x[\cdot] = \sum_{k=-\infty}^{\infty} x[k]\delta[\cdot - k]$$

By linearity, the output is then given by

$$y[\cdot] = \sum_{k=-\infty}^{\infty} x[k]h[\cdot - k]$$

5. For a fixed n , the convolution sum (\spadesuit) involves:

- time-reversing either sequence, e.g., $x[k] \rightarrow \tilde{x}[k] = x[-k]$;
- delaying the time-reversed sequence by n samples: $\tilde{x}[k] \rightarrow \tilde{x}[k - n] = x[n - k]$;
- computing the sum of products $\sum_k h[k]\tilde{x}[k - n] = \sum_k h[k]x[n - k]$.

To determine the entire output sequence $y[\cdot]$, this computation is performed for every value of n .

6. **Example.** Let the input to an FIR filter with coefficient vector $\mathbf{b} = [2 \ -1 \ 1 \ -2]^T$ be given by

$$x[n] = \delta[n] + 2\delta[n - 1] + 3\delta[n - 2] - \delta[n - 3]$$

Clearly, the output $y[n]$ equals zero for $n < 0$ and $n \geq 3 + 4 = 7$, and thus the output sequence has finite duration. The nonzero portion $y[0 : 6]$ of $y[\cdot]$ is computed below.

| k | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | $y[n]$ |
|------------|----|----|----|----|----|----|----|---|---|---|--------|
| $h[k]$ | | | | 2 | -1 | 1 | -2 | | | | |
| $x[-k]$ | -1 | 3 | 2 | 1 | | | | | | | 2 |
| $x[1 - k]$ | | -1 | 3 | 2 | 1 | | | | | | 3 |
| $x[2 - k]$ | | | -1 | 3 | 2 | 1 | | | | | 5 |
| $x[3 - k]$ | | | | -1 | 3 | 2 | 1 | | | | -5 |
| $x[4 - k]$ | | | | | -1 | 3 | 2 | 1 | | | 0 |
| $x[5 - k]$ | | | | | | -1 | 3 | 2 | 1 | | -7 |
| $x[6 - k]$ | | | | | | | -1 | 3 | 2 | 1 | 2 |

7. In general, if \mathbf{h} and \mathbf{x} are finite-duration sequences with nonzero values limited to the *activity intervals* $0 : K - 1$ and $0 : L - 1$ respectively, then $\mathbf{y} = \mathbf{h} * \mathbf{x}$ is also a finite-duration sequence whose nonzero values are limited to the (activity) interval $0 : K + L - 2$. We can thus define the non-circular convolution of two *vectors* \mathbf{b} and \mathbf{s} of length K and L by embedding them into all-zeros sequences, i.e.,

$$h[n] = \begin{cases} b[n], & 0 \leq n \leq K - 1; \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad x[n] = \begin{cases} s[n], & 0 \leq n \leq L - 1; \\ 0, & \text{otherwise} \end{cases}$$

and computing the convolution $\mathbf{h} * \mathbf{x}$ at time indices $0 : K + L - 2$. In other words,

$$\mathbf{b} * \mathbf{s} = y[0 : K + L - 2]$$

From the previous example, we have

$$[2 \ -1 \ 1 \ -2]^T * [1 \ 2 \ 3 \ -1]^T = [2 \ 3 \ 5 \ -5 \ 0 \ -7 \ 2]^T$$

8. Using linearity and/or time invariance, it is easy to show that

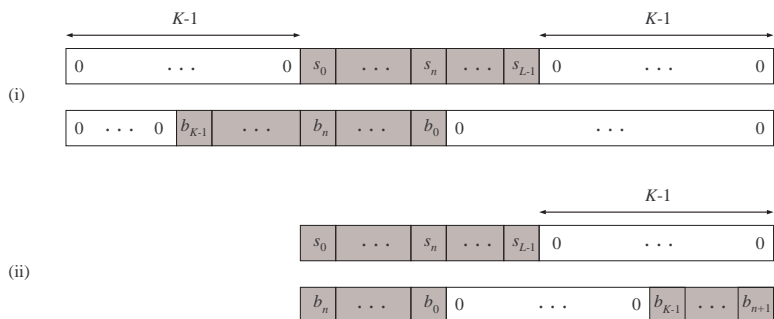
$$\begin{aligned} \mathbf{b} * [\mathbf{0}_i; \mathbf{s}] &= [\mathbf{0}_i; \mathbf{b} * \mathbf{s}] \\ \mathbf{b} * [\mathbf{s}; \mathbf{0}_i] &= [\mathbf{b} * \mathbf{s}; \mathbf{0}_i] \\ \mathbf{b} * (\alpha \mathbf{s}) &= \alpha (\mathbf{b} * \mathbf{s}) \\ \mathbf{b} * (\mathbf{r} + \mathbf{s}) &= \mathbf{b} * \mathbf{r} + \mathbf{b} * \mathbf{s} \end{aligned}$$

where α is a scalar and $\mathbf{0}_i$ is an all-zeros vector of length i .

9. Linear convolution of two vectors \mathbf{b} and \mathbf{s} of length K and L (respectively) produces a vector of length $K + L - 1$. An alternative way of obtaining this is by zero-padding each vector to length $K + L - 1$, followed by a circular convolution of the two zero-padded vectors:

$$\mathbf{b} * \mathbf{s} = [\mathbf{b}; \mathbf{0}_{L-1}] \otimes [\mathbf{s}; \mathbf{0}_{K-1}]$$

This is illustrated in the figure below. The value of the convolution sum is obtained, for n in $0 : K + L - 2$, by taking the dot product of either pair of vectors shown.



- In (i), the shaded portion of the bottom vector is shifted linearly from (flush) left to (flush) right.
- In (ii), the vector $[\mathbf{b}; \mathbf{0}_{L-1}]$ is circularly reversed and shifted to the right (as needed for circular convolution).

10. **Example.** Verify that the vectors $\mathbf{c1}$ and $\mathbf{c2}$ below are identical:

```

b = [2 -1 1 -2 ].' ;
s = [1 2 3 -1 ].' ;
c1 = conv(b,s) ;
B = fft(b,7) ;
S = fft(s,7) ;
c2 = ifft(B.*S) ;

```

11. If a FIR filter is to be implemented using a fixed-length convolution algorithm (e.g., based on DFTs of fixed size), the input sequence must be parsed into consecutive blocks of suitable

length L . Convolution is performed over individual blocks, and the output is obtained using the relationship

$$\begin{aligned}\mathbf{b} * [\mathbf{s}^{(1)}; \mathbf{s}^{(2)}] &= \mathbf{b} * [\mathbf{s}^{(1)}; \mathbf{0}_L] + \mathbf{b} * [\mathbf{0}_L; \mathbf{s}^{(2)}] \\ &= [\mathbf{b} * \mathbf{s}^{(1)}; \mathbf{0}_L] + [\mathbf{0}_L; \mathbf{b} * \mathbf{s}^{(2)}]\end{aligned}$$

and its obvious extensions. This is illustrated in the example below. Note the overlap between the nonzero portions of $[\mathbf{b} * \mathbf{s}^{(1)}; \mathbf{0}_6]$ and $[\mathbf{0}_6; \mathbf{b} * \mathbf{s}^{(2)}]$.

12. Example. Let $\mathbf{b} = [1 \ -3 \ -3 \ 1]^T$ and

$$\mathbf{s} = [1 \ -2 \ 3 \ 4 \ 4 \ -2 \ 7 \ 5 \ 1 \ 3 \ -1 \ -4]^T$$

We divide \mathbf{s} into two blocks of length $L = 6$:

$$\mathbf{s}^{(1)} = [1 \ -2 \ 3 \ 4 \ 4 \ -2]^T \quad \text{and} \quad \mathbf{s}^{(2)} = [7 \ 5 \ 1 \ 3 \ -1 \ -4]^T$$

Using two convolutions, we obtain

$$\begin{aligned}\mathbf{b} * \mathbf{s}^{(1)} &= [1 \ -5 \ 6 \ 2 \ -19 \ -23 \ -2 \ 10 \ -2]^T \\ \mathbf{b} * \mathbf{s}^{(2)} &= [7 \ -16 \ -35 \ -8 \ -8 \ -9 \ 18 \ 11 \ -4]^T\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{b} * \mathbf{s} &= [\mathbf{b} * \mathbf{s}^{(1)}; \mathbf{0}_6] + [\mathbf{0}_6; \mathbf{b} * \mathbf{s}^{(2)}] \\ &= [1 \ -5 \ 6 \ 2 \ -19 \ -23 \ \boxed{5 \ -6 \ -37} \ -8 \ -8 \ -9 \ 18 \ 11 \ -4]^T\end{aligned}$$

LECTURE 24

Topics: convolution in the z -domain; frequency-selective filters

Textbook References: sections 4.5, 4.6.5

Key Points:

- If

$$X(z) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} x[n]z^{-n},$$

then the input $x[\cdot]$, impulse response $h[\cdot]$ and output $y[\cdot]$ of a linear time-invariant system are related by

$$Y(z) = H(z)X(z)$$

Thus convolution in the time domain amounts to multiplication in the (so-called) z -domain.

- Ideal filters have amplitude responses which are piecewise constant functions of ω . These responses are unattainable in practice.
- Ideal filters can be approximated by FIR filters. A longer coefficient vector affords a better approximation to an ideal amplitude response, at the expense of a longer delay between input and output.

Theory and Examples:

1. The z -transform of a sequence $\mathbf{x} = x[\cdot]$ is the power series defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Convergence may be an issue if $x[\cdot]$ has infinite duration. Notably, exponential sequences such as $x[n] = z_0^n$ (for all n) do not have a z -transform except in the trivial case $z_0 = 0$.

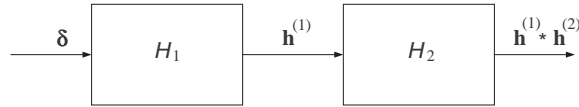
2. The z -transform of a finite-duration sequence is always well defined; it was introduced earlier as the system function of a FIR filter:

$$H(z) = \sum_{n=0}^M b_n z^{-n} = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

We have also seen that the system function $H(z)$ of the cascade of two FIR filters (labeled 1 and 2) satisfies

$$H(z) = H_1(z)H_2(z)$$

This result was obtained by applying a two-sided exponential input to the cascade.



3. By applying an input $\delta = \delta[\cdot]$ to the same cascade, we see (figure above) that the impulse response of the cascade is given by

$$\mathbf{h} = \mathbf{h}^{(1)} * \mathbf{h}^{(2)}$$

Thus convolution of sequences in the time domain is equivalent to multiplication of their z -transforms.

4. The above result:

- can be restated as $\mathbf{y} = \mathbf{h} * \mathbf{x} \Leftrightarrow Y(z) = H(z)X(z)$;
- extends to infinite-duration sequences \mathbf{x} and \mathbf{h} provided both z -transforms converge;
- can be also shown directly by direct multiplication of the power series $H(z)$ and $X(z)$, i.e., the resulting coefficient of z^{-n} is given by $\sum_k h[k]x[n-k]$.

5. **Example.** The result

$$[2 \ -1 \ 1 \ -2]^T * [1 \ 2 \ 3 \ -1]^T = [2 \ 3 \ 5 \ -5 \ 0 \ -7 \ 2]^T$$

can be also obtained through the product

$$(2 - z^{-1} + z^{-2} - 2z^{-3})(1 + 2z^{-1} + 3z^{-2} - z^{-3}) = 2 + 3z^{-1} + 5z^{-2} - 5z^{-3} - 7z^{-5} + 2z^{-6}$$

6. An FIR filter with real-valued coefficients b_0, \dots, b_M has frequency response

$$H(e^{j\omega}) = b_0 + b_1 e^{-j\omega} + \dots + b_M e^{-jM\omega}$$

In cases where the coefficients exhibit odd or even symmetry, the amplitude response $|H(e^{j\omega})|$ can be expressed in terms of either cosines or sines.

In the general (not necessarily symmetric) case, it is also easy to write the *squared* amplitude response

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = H(e^{j\omega})H(e^{-j\omega})$$

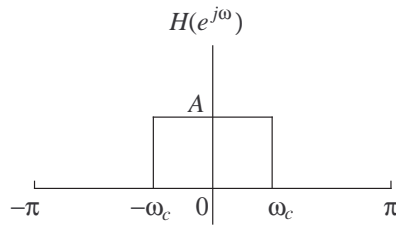
as a sum of finitely many cosines. For example, if $\mathbf{b} = [1 \ 2]^T$, then

$$|H(e^{j\omega})|^2 = (1 + 2e^{-j\omega})(1 + 2e^{j\omega}) = 5 + 4 \cos \omega$$

(**Your task:** verify the last equality.)

7. A finite sum of cosines of different frequencies (i.e., multiples of ω) is a smooth curve that has no discontinuities or flat sections. This means that the filter with real-valued frequency response $H(e^{j\omega})$ shown below (for $\omega \in (-\pi, \pi]$) *cannot* be an FIR filter.

This filter is known as an *ideal lowpass filter* of unit gain and zero delay. Ideal frequency-selective filters are characterized by piecewise constant amplitude response and linear phase response (for the filter shown above, the phase response equals zero at all frequencies). Using arguments beyond the scope of this course, it can be shown that such filters cannot be synthesized in practice, whether in FIR form or as any other filter structure.



8. FIR filters with characteristics which approximate those of ideal filters can be obtained using numerical algorithms. The amplitude (or magnitude) response of a practical lowpass filter with real coefficients is symmetric about $\omega = 0$ and has the following features:

- Its value varies between $A(1 - \delta)$ and $A(1 + \delta)$ in the passband $[0, \omega_p]$; the factor δ is the passband ripple.
- It has a maximum value of $A\epsilon$ in the stopband $[\omega_s, \pi]$; the factor $1/\epsilon$ is the stopband attenuation.
- Its value drops from $A(1 - \delta)$ to $A\epsilon$ over the transition band $[\omega_p, \omega_s]$; thus the point ω_c (cutoff frequency) in the ideal response is approximated by an interval.

9. The MATLAB filter design and analysis tool (FDATOOL) is a comprehensive interface for designing digital filters of various types. These designs can be also obtained using MATLAB command-line functions. The function `FIRPM` is used for FIR filters. As an example,

```
b = firpm( M, [0.0 fp fs 0.5]*2, [A A 0 0] );
```

produces a lowpass FIR filter with coefficient vector \mathbf{b} of length $M + 1$. The passband and stopband edges are $\omega_p = 2\pi f_p$ and $\omega_s = 2\pi f_s$, respectively. The ideal passband and stopband gains are A and 0 , respectively. (Additional parameters can be inserted to control the relative amounts of passband ripple and stopband attenuation.)

Your task: Run

```
b = firpm( 36, [0.0 0.12 0.15 0.5]*2, [1 1 0 0] );
```

and plot the amplitude and phase response of the filter. Determine the passband ripple δ and stopband attenuation $1/\epsilon$.

10. Approximations to a piecewise constant amplitude response can be obtained using symmetric coefficient vectors, i.e., such that $b_n = b_{M-n}$. As we saw in a previous reading assignment, this results in

$$H(e^{j\omega}) = e^{-j\omega M/2} F(\omega)$$

where $F(\omega)$ is a real-valued sum of cosines. The amplitude response is $|F(\omega)|$, while the frequency response is

$$\angle H(e^{j\omega}) = -\frac{M\omega}{2} + (0 \text{ or } \pi)$$

The phase response over the passband has no discontinuities, and is simply given by $-M\omega/2$. This amounts to a uniform delay of $M/2$ samples for all sinusoidal components in the passband.

Larger values of M allow for a better approximation to the ideal response (as there are more parameters available to shape $F(\omega)$); however, they also lead to longer delays between input and output. This is illustrated in the example below.

- 11. Example.** The delay between input and output is evident in the following example, which uses the filter designed earlier. Run step by step.

```
n = (1:500).';
s = exp(-((n-200).^2)/100^2) ;
plot(n,s) ;           % slowly varying signal
%
% add noise in the stopband:
%
z = 0.3*cos(0.42*pi*n) + 0.1*cos(0.53*pi*n);
x = s+z ;
plot(n,x)
y = filter(b,1,x) ;   % run filter on x
plot(n,y)
hold
plot(n,s)             % note delay
```

- 12.** A lowpass filter can be easily transformed to a highpass filter, and vice versa. In the frequency domain, this amounts to a frequency shift by π radians per sample. In the time domain, this is accomplished by the transformation

$$\tilde{b}_n = (-1)^n b_n, \quad n \in \mathbf{Z}$$

i.e., multiplying each entry of \mathbf{b} by the sinusoid of frequency π . Indeed,

$$\tilde{H}(e^{j\omega}) = \sum_{n=0}^M \tilde{b}_n e^{-j\omega n} = \sum_{n=0}^M b_n e^{j\pi n} e^{-j\omega n} = H(e^{j(\omega-\pi)})$$