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- $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle^*$

1. If

$$\mathbf{u} = \begin{bmatrix} 5 & j & 1-j \end{bmatrix}^T \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 1 & 2+j & 3 \end{bmatrix}^T ,$$

then the inner product $\langle \mathbf{u}, \mathbf{v} \rangle$ equals

A. $7+j$

B. $7-j$

C. $9+j$

D. $9-j$

2. If

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then $\|\mathbf{u}+\mathbf{v}\|^2$ equals

A. 43

B. 51

C. 61

D. 197

3. Vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually orthogonal, each having norm equal to 2. If

$$\begin{aligned} \mathbf{x} &= \mathbf{u} + 2j\mathbf{v} - 3\mathbf{w} \\ \mathbf{y} &= (1+j)\mathbf{u} - \mathbf{v} + \mathbf{w}\ , \end{aligned}$$

then $\langle \mathbf{x}, \mathbf{y} \rangle$ equals

A. $-2+3j$

B. $-2-j$

C. $-8+12j$

D. $-8-4j$

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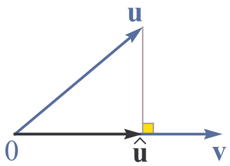

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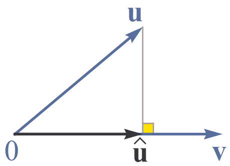

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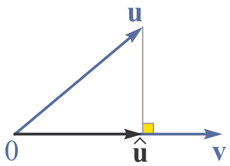

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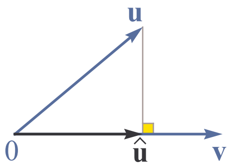

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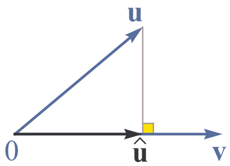

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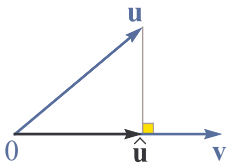

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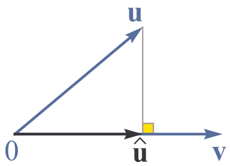

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where each c_k = projection coefficient of $\mathbf{s}$ onto the k^{th} column of $\mathbf{V}$.

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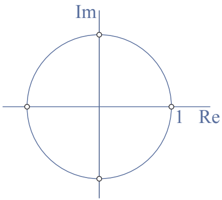

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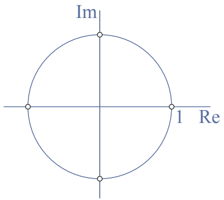

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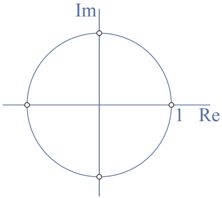

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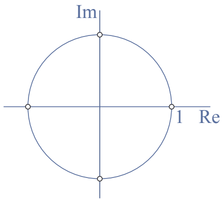

- $\mathbf{V}^T = \mathbf{V}$
- $\mathbf{V}^H\mathbf{V} = 4\mathbf{I}$ (columns are orthogonal, each has norm $= 2$)

4. If

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix},$$

then $4 \cdot c_3$ equals

A. $s_0 + s_1 + s_2 + s_3$

B. $s_0 + js_1 - s_2 - js_3$

C. $s_0 - s_1 + s_2 - s_3$

D. $s_0 - js_1 - s_2 + js_3$

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$m - n > 0$ columns of $\mathbf{V}$ deleted $\Rightarrow$ no longer possible to express every $\mathbf{s} \in \mathbb{C}^m$ as a linear combination $\mathbf{V}\mathbf{c}$ of the remaining columns $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}$

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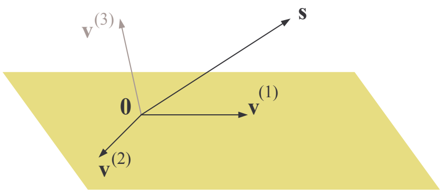

Linear Least Squares Approximation

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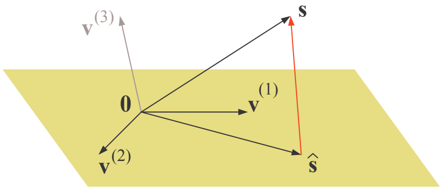

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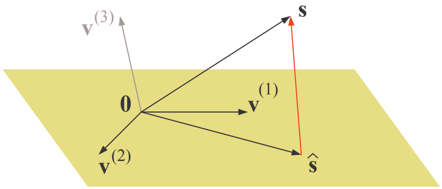

Problem: Approximate $\mathbf{s}$

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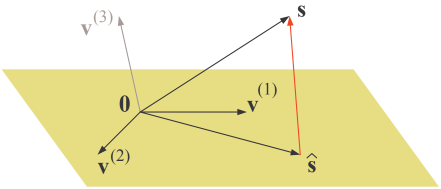

Problem: Approximate $\mathbf{s}$ by $\hat{\mathbf{s}} \;=\; \mathbf{V}\mathbf{c}$

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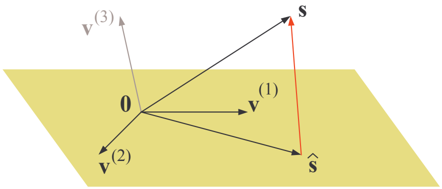

Problem: Approximate $\mathbf{s}$ by $\hat{\mathbf{s}} = \mathbf{V}\mathbf{c}$ so as to minimize the error vector norm $\|\mathbf{s} - \hat{\mathbf{s}}\|$.

Linear Least Squares Approximation

If the $m \times m$ matrix $\mathbf{V}$ is invertible, the equation

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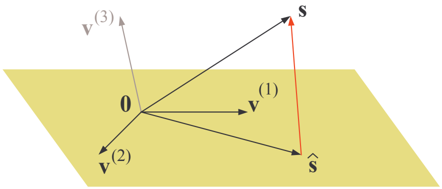

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Solution:

Linear Least Squares Approximation

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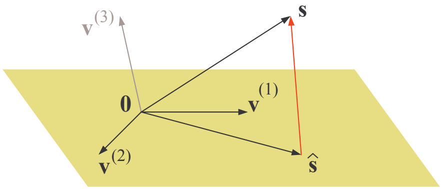

Problem: Approximate $\mathbf{s}$ by $\hat{\mathbf{s}} = \mathbf{V}\mathbf{c}$ so as to minimize the error vector norm $\|\mathbf{s} - \hat{\mathbf{s}}\|$.

Solution: $\hat{\mathbf{s}}$ is the orthogonal projection of $\mathbf{s}$ onto

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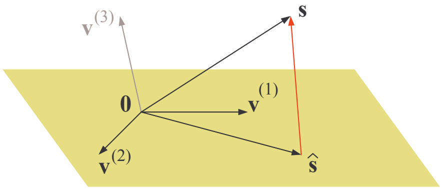
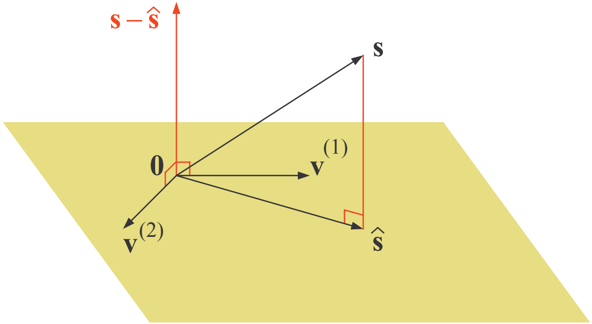


Problem: Approximate $\mathbf{s}$ by $\hat{\mathbf{s}} = \mathbf{V}\mathbf{c}$ so as to minimize the error vector norm $\|\mathbf{s} - \hat{\mathbf{s}}\|$.

Solution: $\hat{\mathbf{s}}$ is the orthogonal projection of $\mathbf{s}$ onto , defined by

$$\mathbf{s} - \hat{\mathbf{s}} \;\perp\; \mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}$$

Linear Least Squares Approximation $\equiv$ Orthogonal Projection



$$\mathbf{s} - \hat{\mathbf{s}} \;\perp\; \mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}$$

The Orthogonal Case

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Let $\mathbf{V}$ be square $(m \times m)$ with orthogonal columns.

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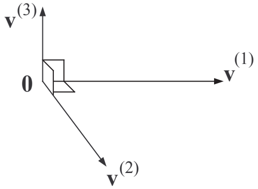

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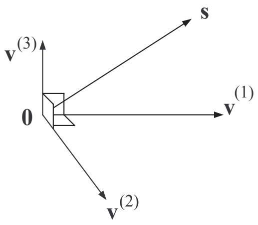
Let $\mathbf{V}$ be square $(m \times m)$ with orthogonal columns.

Then any $\mathbf{s} \in \mathbb{R}^m$



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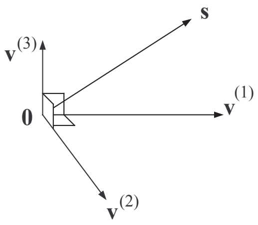
Let $\mathbf{V}$ be square $(m \times m)$ with orthogonal columns.

Then any $\mathbf{s} \in \mathbb{R}^m$ can be represented as

$$\mathbf{s} \; = \; \mathbf{V}\mathbf{c}$$



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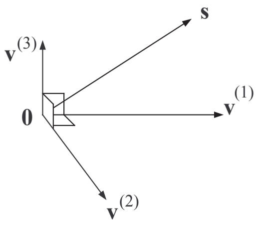

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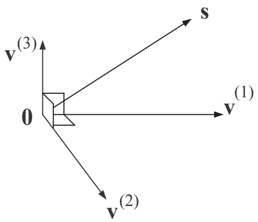
Then any $\mathbf{s} \in \mathbb{R}^m$ can be represented as

$$\mathbf{s} = \mathbf{V}\mathbf{c}, \qquad \text{where} \qquad c_k = \langle \mathbf{v}^{(k)}, \mathbf{s} \rangle / \|\mathbf{v}^{(k)}\|^2$$



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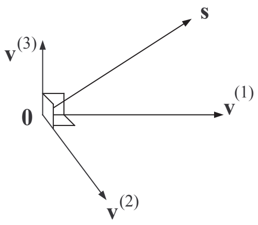

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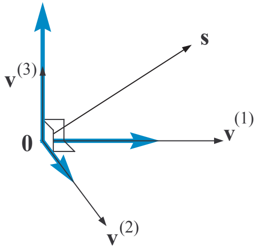

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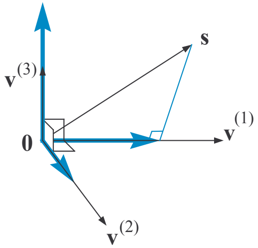

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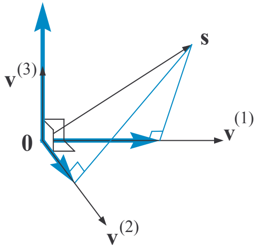

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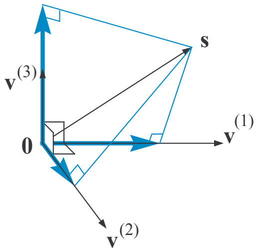

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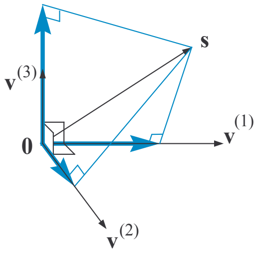

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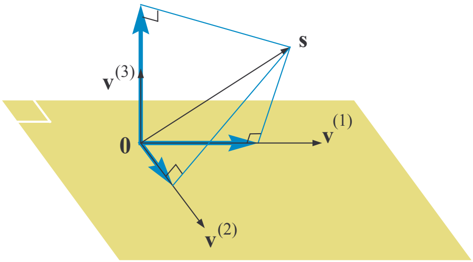

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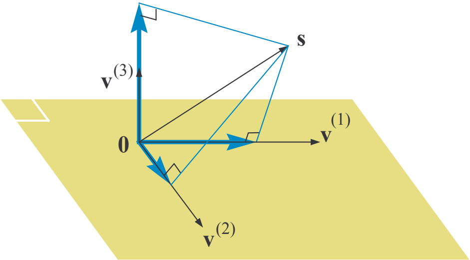

$$\mathbf{s} \;=\; \sum_{k=1}^{n} c_k \mathbf{v}^{(k)} \;+\; \sum_{k=n+1}^{m} c_k \mathbf{v}^{(k)}$$

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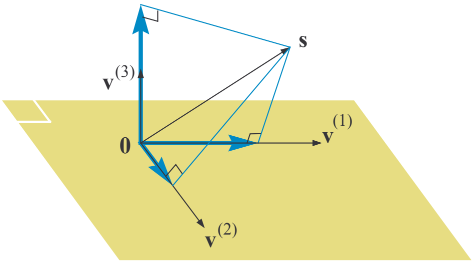

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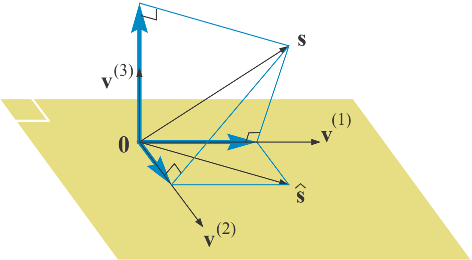

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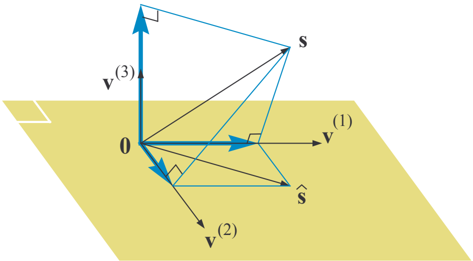

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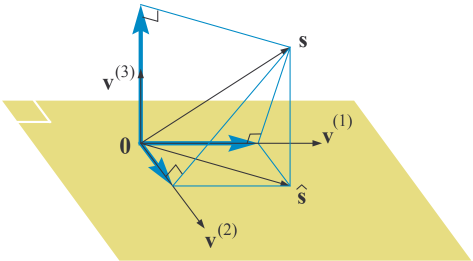

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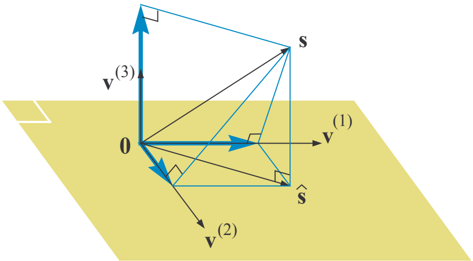

$$\mathbf{s} \;=\; \underbrace{\sum_{k=1}^{n} c_k \mathbf{v}^{(k)}}_{\hat{\mathbf{s}}} \;\underset{\perp}{+}\; \underbrace{\sum_{k=n+1}^{m} c_k \mathbf{v}^{(k)}}_{\mathbf{s} - \hat{\mathbf{s}}}$$

5. Vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually orthogonal and such that $\|\mathbf{u}\| = 1$, $\|\mathbf{v}\| = 2$ and $\|\mathbf{w}\| = 3$. Let

$$\mathbf{s} \ = \ 5\mathbf{u} + 7\mathbf{v} - 2\mathbf{w}$$

If $\hat{\mathbf{s}}$ is the projection of $\mathbf{s}$ onto the subspace defined by $\mathbf{u}$ and $\mathbf{v}$, then $\|\mathbf{s} - \hat{\mathbf{s}}\|$ equals

A. 6

B. 12

C. 18

D. -6

6 . Let

$$\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 6 \\ 1-2j \\ -1 \\ 1+2j \end{bmatrix}$$

If $\hat{\mathbf{x}}$ has the form $\begin{bmatrix} a & a & a & a \end{bmatrix}^T$, then the minimum value of $\|\mathbf{x} - \hat{\mathbf{x}}\|^2$, obtained by suitable choice of a, equals

A. 0

B. 11

C. 44

D. 144