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# Angles and Projection

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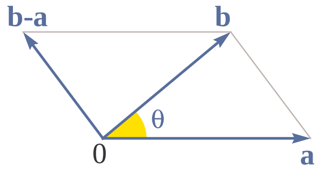

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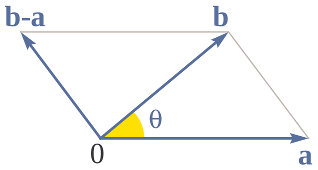


- Expand $\|\mathbf{a} - \mathbf{b}\|^2$

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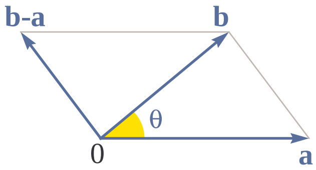


- Expand $\|\mathbf{a} - \mathbf{b}\|^2$ and use the law of cosines

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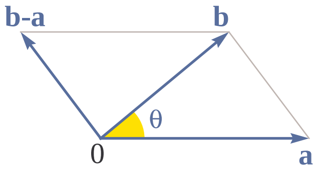


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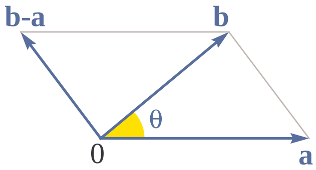


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$$\cos\theta \;=\; \frac{\langle \mathbf{a}, \mathbf{b}\rangle}{\|\mathbf{a}\|\|\mathbf{b}\|}$$

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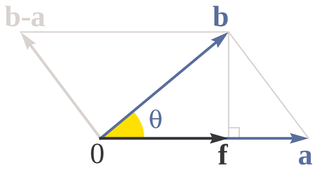


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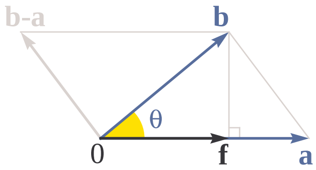


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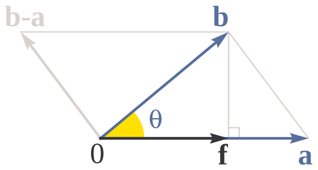


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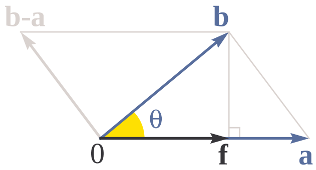


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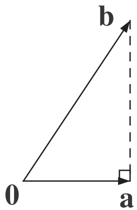
$$\lambda = \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{\|\mathbf{a}\|^2}$$

1 . If vectors $\mathbf{a}$ and $\mathbf{b}$ are as shown in the figure below, which (one or more) of the following statements are true?




A. Vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

B. Vectors $\mathbf{a}$ and $\mathbf{b} - \mathbf{a}$ are orthogonal.

C. $\langle \mathbf{a}, \mathbf{b} \rangle = \|\mathbf{a}\| \cdot \|\mathbf{b}\|$

D. $\langle \mathbf{a}, \mathbf{b} \rangle = \|\mathbf{a}\|^2$

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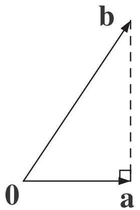


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2 . The angle between $\mathbf{a} = [3 \quad 1 \quad 1 \quad 5]^T$ and $\mathbf{b} = [1 \quad -5 \quad -1 \quad -3]^T$ equals

A. $\pi/3$

B. $\cos^{-1}(-1/8)$

C. $\cos^{-1}(-1/72)$

D. $2\pi/3$

**3 .** The projection of $\mathbf{b} = [12 \quad 3 \ -9 \ -4]^T$ onto $\mathbf{a} = [-1 \quad 2 \quad 4 \quad 2]^T$ is the vector

A. $\mathbf{b} - \mathbf{a}$

B. $\mathbf{a}$

C. $2\mathbf{a}$

D. $-2\mathbf{a}$

**4 .** Vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually orthogonal and such that $\|\mathbf{u}\| = 1$, $\|\mathbf{v}\| = 2$ and $\|\mathbf{w}\| = 3$. If

$$\mathbf{s} \;=\; 3\mathbf{u} - 2\mathbf{v} + \mathbf{w} \;,$$

then $\|\mathbf{s}\|^2 \;=$

A. $10$

B. $14$

C. $22$

D. $34$

**5 .** Vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ each have length $= 2$, and the angle between any two of them equals $\pi/3$ rad. If

$$\mathbf{x} = \mathbf{u} + \mathbf{v} + \mathbf{w} ,$$

then $\|\mathbf{x}\|^2 =$

A. $6$

B. $12$

C. $24$

D. $48$

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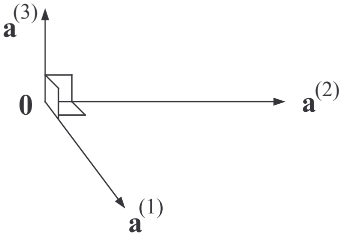

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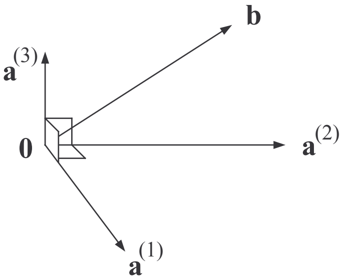

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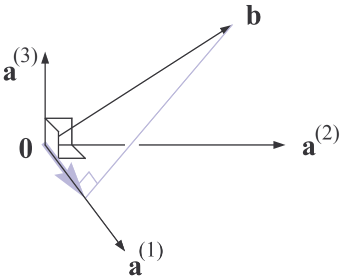

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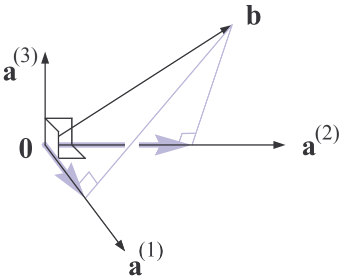

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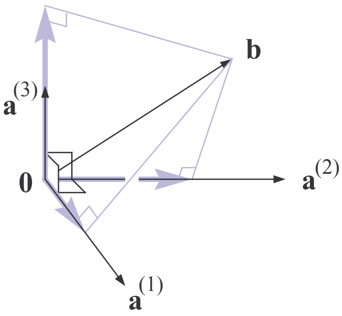

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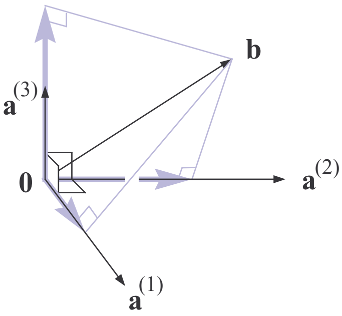


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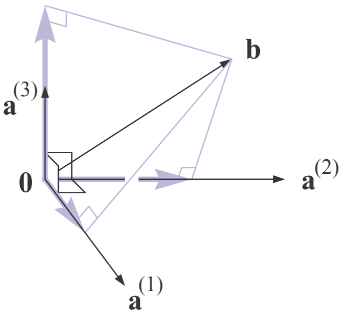


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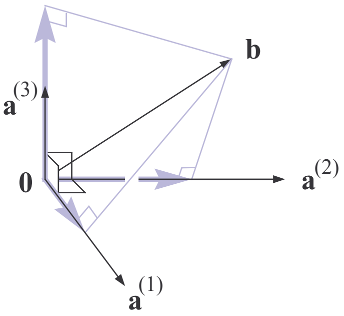


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$$(\forall k) \qquad x_k = \frac{\langle \mathbf{a}^{(k)}, \mathbf{b} \rangle}{\|\mathbf{a}^{(k)}\|^2}$$

**6 .** If

$$\begin{bmatrix} 1 & 2 & 3 & -4 \\ 1 & 2 & -3 & 4 \\ 1 & -2 & 3 & 4 \\ 1 & -2 & -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix},$$

then $x_2$ equals

A. $\dfrac{b_1 + 2b_2 + 3b_3 - 4b_4}{2}$

B. $\dfrac{b_1 + 2b_2 - 3b_3 + 4b_4}{4}$

C. $\dfrac{b_1 + b_2 - b_3 - b_4}{8}$

D. $\dfrac{b_1 + b_2 - b_3 - b_4}{4}$