# Invertible Matrices

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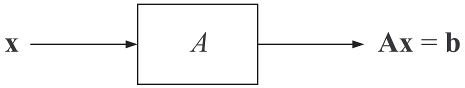

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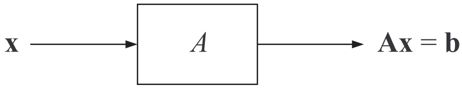


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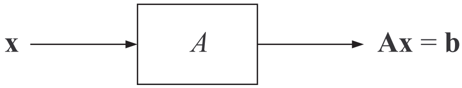


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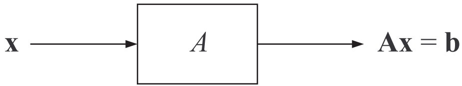


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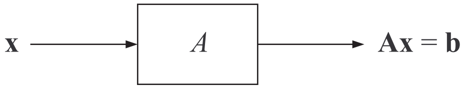


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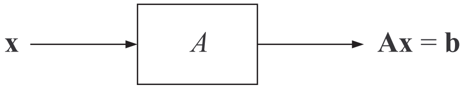


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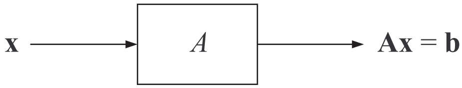


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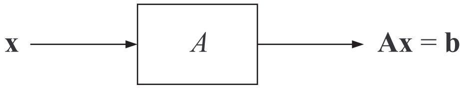


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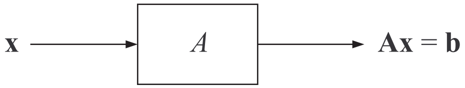


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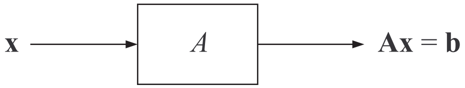


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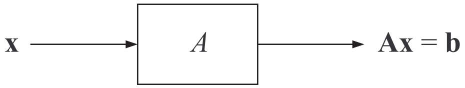


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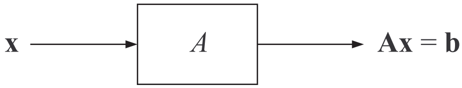


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Only square matrices $(m = n)$ can be invertible.

**1.** Which (one ore more) of the following statements about the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}$$

are correct?

(i) The equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{b} = [\,1 \;\; 0\,]^T$, has a unique solution.

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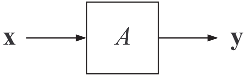

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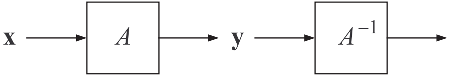

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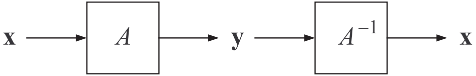

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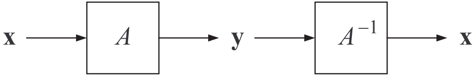


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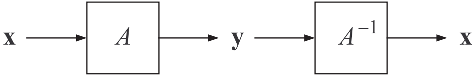


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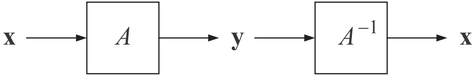


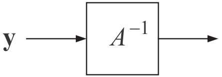


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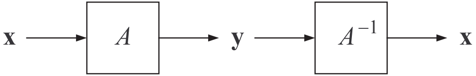


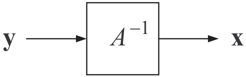


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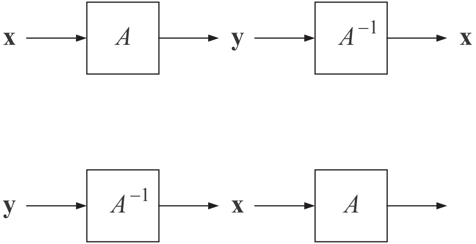


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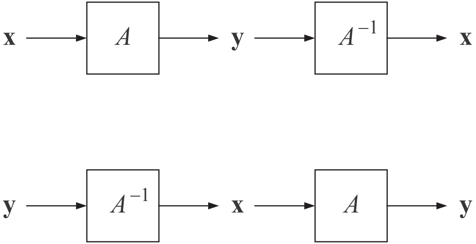


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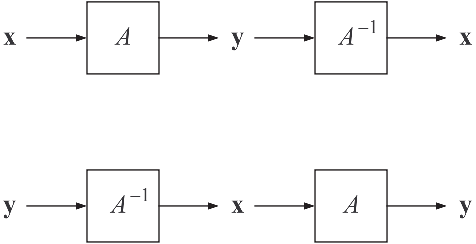


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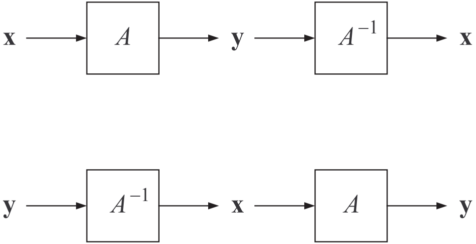


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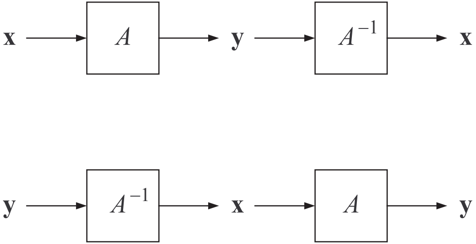


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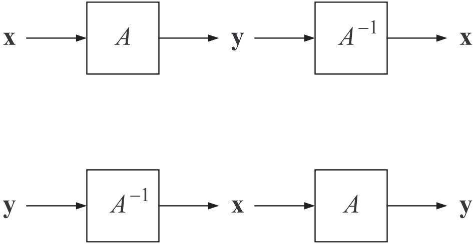


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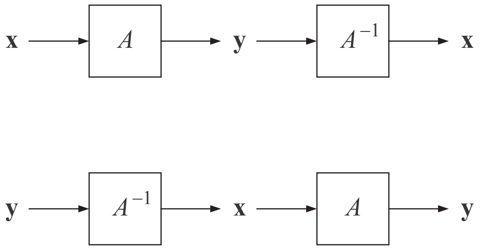


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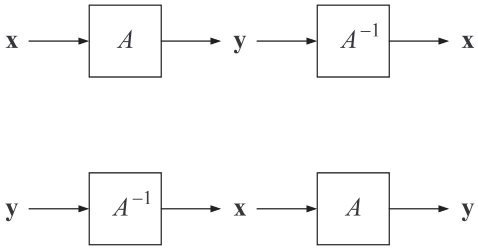


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- $\mathbf{A}\mathbf{B} = \mathbf{I} \quad \Leftrightarrow \quad$ $\mathbf{A}$ and $\mathbf{B}$ are inverses of each other

2. The matrix

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

represents counterclockwise rotation by angle $\theta$ on the Cartesian plane.

Determine $\mathbf{A}^{-1}$.

3. If

$$\mathbf{A}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{A}\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

then $\mathbf{A}^{-1} =$

**4.** Given

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ c & 2 & 1 & 1 \end{bmatrix},$$

what is the value of $c$?

(i) $0$ (ii) $3$ (iii) $4$ (iv) $-3$

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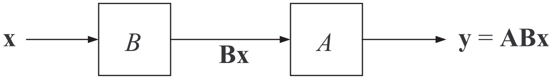

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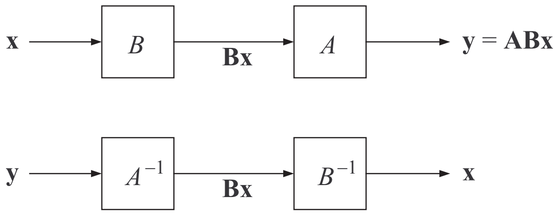

**5.**

If $\begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix}^{-1} = \begin{bmatrix} p & 0 & 0 \\ q & s & 0 \\ r & t & u \end{bmatrix}$, then

$$\begin{bmatrix} f & 0 & 0 \\ c & b & a \\ e & d & 0 \end{bmatrix}^{-1} =$$

# Triangular Matrices

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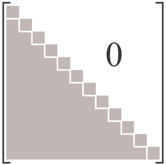
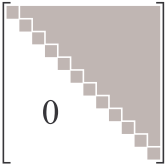
$\mathbf{L} =$ 


$\mathbf{U} =$ 

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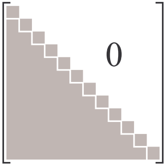
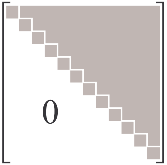
$\mathbf{L} =$ 


$\mathbf{U} =$ 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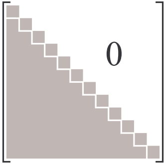
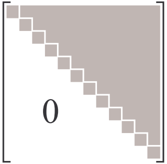
$\mathbf{L} =$ 


$\mathbf{U} =$ 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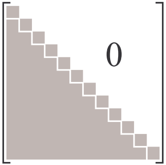
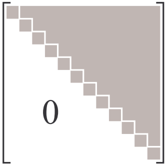
$\mathbf{L} =$ 


$\mathbf{U} =$ 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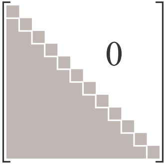
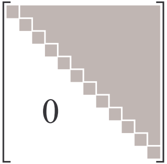
$\mathbf{L} =$ 


$\mathbf{U} =$ 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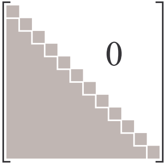
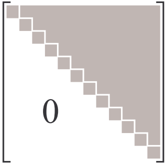
$\mathbf{L} =$ 


$\mathbf{U} =$ 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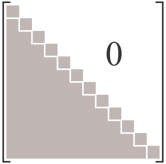
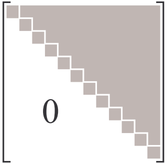
$\mathbf{L} =$ 


$\mathbf{U} =$ 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- Backward substitution solves $\mathbf{Ux} = \mathbf{c}$.

Invertibility (or not) of $\mathbf{A}$ is established during forward elimination.

**6 and 7.** Forward phase of Gaussian elimination:

| $m$ | $x_1$ | $x_2$ | $x_3$ | $b$ |
|---|---|---|---|---|
| | $\boxed{4}$ | 5 | 2 | 12 |
| $-1$ | 4 | 7 | $-2$ | 4 |
| $c$ | 3 | 3 | 2 | 9 |
| | 4 | 5 | 2 | 12 |
| | 0 | $\boxed{2}$ | $-4$ | $-8$ |
| $3/8$ | 0 | $-3/4$ | $1/2$ | 0 |

Determine the value of $c$ and the solution $(x_1, x_2, x_3)$.