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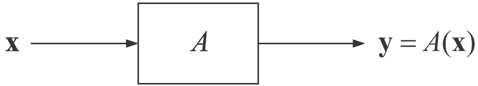

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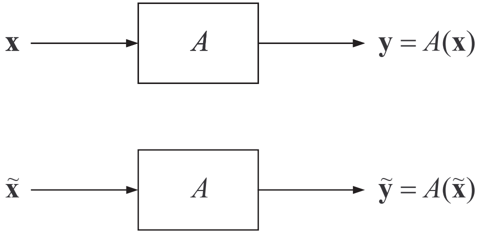

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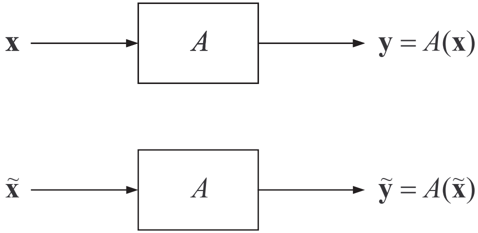

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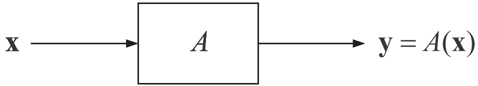

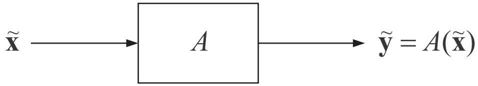

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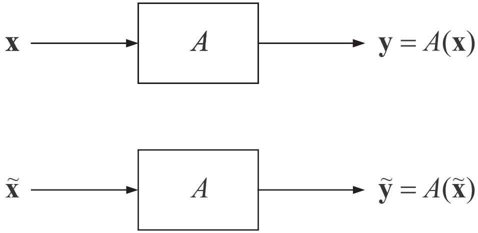

then for any scaling factors α and β,

$$A(\alpha \mathbf{x} + \beta \tilde{\mathbf{x}}) \;=\; \alpha A(\mathbf{x}) + \beta A(\tilde{\mathbf{x}})$$

1. Which (one or more) of the following relationships between $\mathbf{x} = (x_1, x_2, x_3)$ and y represents a linear transformation $\mathbf{R}^3 \to \mathbf{R}$?

A. $y = x_1 + x_2 + x_3$

B. $$y = \begin{cases} x_1, & x_1 \geq 0 \\ 0, & x_1 < 0 \end{cases}$$

C. y equals the length of the vector $\mathbf{x}$

D. y equals the cosine of the angle between the vector $\mathbf{x}$ and the vector $(1, 1, 1)$

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In expressions involving products of matrices and vectors,

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In expressions involving products of matrices and vectors,

- $\mathbf{a}$ will always be a column vector

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$$e_i^{(k)} \;=\; \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$$

Matrix of a Linear Transformation $A(\,\cdot\,)$

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$$A : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

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$$\mathbf{x} \;=\; x_1\mathbf{e}^{(1)} + x_2\mathbf{e}^{(2)} + \cdots + x_n\mathbf{e}^{(n)}$$

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Then by linearity,

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$$A : \mathbb{R}^n \to \mathbb{R}^m$$

Express the input vector as

$$\mathbf{x} \;=\; x_1 \mathbf{e}^{(1)} + x_2 \mathbf{e}^{(2)} + \cdots + x_n \mathbf{e}^{(n)}$$

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Matrix of $A(\cdot)$:

$$\mathbf{A} \;=\; \left[\; A(\mathbf{e}^{(1)}) \quad A(\mathbf{e}^{(2)}) \quad \ldots \quad A(\mathbf{e}^{(n)}) \;\right]$$

$$\mathbf{A} \;\in\; \mathbb{R}^{m \times n}$$

2. Let A be the linear transformation $\mathbf{R}^3 \to \mathbf{R}^3$ such that

$$A(x_1, x_2, x_3) \ = \ (x_3, 2x_1, x_2)$$

What is the matrix $\mathbf{A}$ of this transformation?

Matrix-Vector Product

Matrix-Vector Product

Every $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}^{(1)} & \mathbf{a}^{(2)} & \ldots & \mathbf{a}^{(n)} \end{bmatrix}$$

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For $\mathbf{x} \in \mathbb{R}^{n \times 1}$,

$$\mathbf{Ax} \stackrel{\text{def}}{=} A(\mathbf{x}) = x_1\mathbf{a}^{(1)} + x_2\mathbf{a}^{(2)} + \cdots + x_n\mathbf{a}^{(n)}$$

Equivalently:

Matrix-Vector Product

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Equivalently:

dot product

$$(\mathbf{Ax})_i \;=\; (i^{\text{th}} \text{ row of } \mathbf{A}) \bullet \mathbf{x}$$

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Equivalently:

dot product

$$(\mathbf{Ax})_i \;=\; (i^{\text{th}} \text{ row of } \mathbf{A}) \bullet \mathbf{x} \;=\; \sum_{j=1}^{n} a_{ij}x_j$$

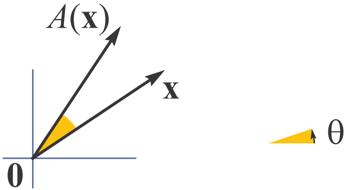
4. If

$$\mathbf{A} = \begin{bmatrix} 5 & -2 & 1 \\ 3 & 0 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix},$$

then $\mathbf{Ax} =$

Rotation Matrix

Rotation Matrix



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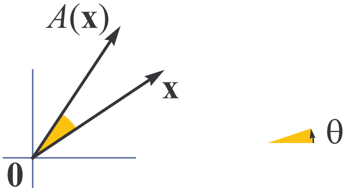

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

3. Which of the following transformations $\mathbf{R}^2 \to \mathbf{R}^2$, described in geometric terms, does the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

represent?

A. Projection of a point on the horizontal (x_1) axis

B. Projection of a point on the vertical (x_2) axis

C. Counterclockwise rotation of a vector (from the origin to a point) by $\pi/2$ radians

D. Reflection of a point about the straight line which bisects the first and third quadrants

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5. If

$$\mathbf{A}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \mathbf{u} \qquad \text{and} \qquad \mathbf{A}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \mathbf{v}\ ,$$

then the first column of $\mathbf{A}$ is given by the vector

A. $\mathbf{u} + \mathbf{v}$

B. $\mathbf{u} - \mathbf{v}$

C. $2\mathbf{u} + 3\mathbf{v}$

D. $\mathbf{u} - 2\mathbf{v}$

6. If

$$\mathbf{A}\begin{bmatrix}1\\2\\0\end{bmatrix}=\begin{bmatrix}2\\1\\5\end{bmatrix} \quad \text{and} \quad \mathbf{A}\begin{bmatrix}-1\\1\\0\end{bmatrix}=\begin{bmatrix}1\\2\\1\end{bmatrix},$$

which (one or more) of the following statements are correct?

A. The dimensions of $\mathbf{A}$ cannot be determined based on the given information.

B. $\mathbf{A}$ is a 3×3 matrix.

C. The second column of $\mathbf{A}$ is given by $[\,1 \;\; 1 \;\; 2\,]^T$.

D. The entries of the third column of $\mathbf{A}$ cannot be determined based on the given information.

6. If

$$\mathbf{A}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \qquad \text{and} \qquad \mathbf{A}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

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See also example in L7.2 7