Results on codes with few distances

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D. Ray-Chaudhuri and R. Wilson (1975):

Let $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be a family of $w$-subsets of $[n]$ such that $|F_i \cap F_j| \in \{p_1, p_2, \ldots, p_s\}$, $i \neq j$

Then $m \leq \binom{n}{s}$

Consider $\mathcal{F} = \{\text{all } w\text{-subsets of } [n]\}$, then $s = w$, so this estimate is tight


Let $C \subset S^{n-1}$ be a set of points $(x_1, x_2, \ldots, x_m)$ such that $(x_i, x_j) \in \{t_1, t_2, \ldots, t_s\}$, $i \neq j$

Then $m \leq \binom{n + s - 1}{s} + \binom{n + s - 2}{s - 1}$

$s = 1$: the bound $m \leq n$ is attained by the $C = \{n \text{ vertices of a regular simplex}\}$

In this talk we will provide a general improvement of these theorems, together with some other results.
Some improvements are known:

e.g., take $X=S^{n-1}$, $s=2$, then the bound gives $|C| \leq \frac{n(n+3)}{2}$

Take $e_1, \ldots, e_{n+1}$ to be the standard basis;

$C = \{e_i + e_j, 1 \leq i < j \leq n+1\}$

Scalar products 1 or 0, so $C$ is a spherical 2-distance set

$|C| = \frac{n(n+1)}{2}$

Larger codes with distances $t_1, t_2$ such that $t_1 + t_2 \geq 0$ do not exist since in this case the DGS bound can be improved to

$m \leq \frac{n(n+1)}{2}$

(Musin, 2008)
Harmonic analysis on homogeneous spaces

X compact metric space, \( d(.,.) \) - distance

Harmonic analysis on \( X \):
Let \( G \) be the isometry group of \( X \) which acts transitively on it
\( H \) -- subgroup that fixes a point \( x_0 \)
\( X \cong \) set of (right) cosets of \( H \) in \( G \)

\[
L_2(G,d\mu) = \bigoplus_{i=0}^{V_i}
\]

\( V_i = (\varphi_{i,1}, \ldots, \varphi_{i,h_i}) \)

There is a unique \( H \)-invariant function \( \Phi_i \)
\( \Phi_i(x,y) = \sum_{j=1}^{h_i} \varphi_{i,j}(x) \varphi_{i,j}(y) \)
\( \Phi_0 = 1 \)

\( G \) acts distance-transitively:
\( p_i(d(x,y)) = \Phi_i(x,y) \)
\( p_0(x) = 1 \)
\( xp_i(x) = a_i p_{i+1}(x) + b_i p_i(x) + c_i p_{i-1}(x) \)

\[
K_i(d(x,y)) = \sum_{j=1}^{h_i} \varphi_{i,j}(x) \varphi_{i,j}(y)
\]

\( K_0(x) = 1, \quad K_1(x) = n-2x \)
\( K_2(x) = 2x^2 - 2nx + \frac{1}{2}n(n-1), \ldots \)
\( xK_i = -((i+1)/2)K_{i+1} + (n/2)K_i - ((n-i+1)/2)K_{i+1} \)

**Theorem** (Delsarte; DGS, 73-77): Let \( C \subset X \) be a code with \( s \) distances (s-code). Then

\[
|C| \leq h_0 + h_1 + \ldots + h_s
\]

Later proofs of particular cases and generalizations obtained by
Frankl-Wilson ('85), Alon-Babai-Suzuki ('91), Babai-Snevily-Wilson ('95) and others.
Parameters of metric space of coding theory

<table>
<thead>
<tr>
<th>$X$</th>
<th>$H_n$</th>
<th>$J^{n,w}$</th>
<th>$S^n$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{\mu}$</td>
<td>$2^{-n} \binom{n}{i}$</td>
<td>$\frac{(w)^i (n-w)^{n-i}}{n}$</td>
<td>$\frac{n-2}{\omega_n} (1 - x^2)^{n-3}/2 , dx$</td>
<td></td>
</tr>
<tr>
<td>$h_i$</td>
<td>$\binom{n}{i}$</td>
<td>$\frac{\binom{n}{n-1}}{i}$</td>
<td>$\binom{n+i-2}{i} + \binom{n+i-3}{i-1}$</td>
<td>$\frac{n-2+i}{n-2+2i}$</td>
</tr>
<tr>
<td>$a_i$</td>
<td>$-\frac{i+1}{2}$</td>
<td>$\frac{(n+1)(w-i)(n-w-i)}{(n-2i-1)(n-2i)}$</td>
<td>$0$</td>
<td>$i$</td>
</tr>
<tr>
<td>$b_i$</td>
<td>$\frac{n}{n-2i+2}$</td>
<td>$\frac{(n+2)(w(n-w)-ni(n-i+1)}{(n-2i)(n-2i+2))}$</td>
<td>$\frac{n-2+i}{n-2+2i}$</td>
<td></td>
</tr>
<tr>
<td>$c_i$</td>
<td>$-\frac{n-i+1}{2}$</td>
<td>$\frac{(w-i+1)(n-w-i+1)(n-i+2)}{(n-2i+2)(n-2i+3)}$</td>
<td>$0$</td>
<td>$i$</td>
</tr>
</tbody>
</table>

Hahn polynomials:

\[
Q_i(x) = \left(\binom{n}{i} - \binom{n}{i-1}\right) \sum_{j=0}^{i} (-1)^j \frac{\binom{i}{j} \binom{n+1-i}{j}}{\binom{w}{j} \binom{n-w}{j}} x^j.
\]
Theorem 1  Let $C \subset X$ be an $s$-code with distances $d_1, \ldots, d_s$. Consider the polynomial $f(x) = \prod_{i=1}^{s}(d_i - x)$ and suppose that its expansion in the basis $\{p_i\}$ has the form $f(x) = \sum_i f_ip_i(x)$. Then

$$|C| \leq \sum_{i: f_i > 0} h_i.$$

Proof: Consider the matrix $H_l$ given by $(H_l)_{i,j} = \phi_{l,j}(x_i)$. Let $\mathcal{H} = (H_1, \ldots, H_s)$ and

$$F = \text{diag}(f_0, f_1, \ldots, f_1, \ldots, f_s, \ldots, f_s).$$

$$A_{p,r} := \mathcal{H} F \mathcal{H}^t = \sum_{i=0}^{s} f_i \sum_{j=1}^{h_i} \phi_{i,j}(x_p) \phi_{i,j}(x_r) = \sum_{i} f_i p_i(d(x_p, x_r))$$

$$= f(d(x_p, x_r)) = \delta_{p,r} f(0), \quad 1 \leq p, r \leq |C|,$$

so $A = f(0)I_{|C|}$.

We have found $|C|$ vectors $z_i = (((H_l)_{i,j}, j = 1, \ldots, \sum_{p=0}^{s} h_p) that lie in the positive subspace of the quadratic form $F$ and satisfy $(Fz_u, z_v) = 0$ for $u \neq v$.

It follows that the $z_i$ are linearly independent

$\Rightarrow$ their number $\leq$ the number of positive eigenvalues of $F$
Polynomials \{p_i, \ i=0,1,…\}

\[ L(g) = \int g \, d\mu \] – linear functional that controls orthogonality

\[ r_i \, L(p_i \, p_j) = \delta_{ij} \quad r_i = 1/h_i \]

\[ L(x^k \, p_m) = 0 \] for any \( k < m \)

\( p_0(x) = 1 \)

\[ x p_i(x) = a_i p_{i+1}(x) + b_i p_i(x) + c_i p_{i-1}(x) \]

**Lemma:** Consider the polynomial \( f(x) = \prod_{i=1}^s (d_i-x) = \sum_{i=0}^s f_i p_i(x) \). Then

\[ f_s = (-1)^s r_s \, c_1 c_2 … c_s \]

\[ f_{s-1} = (-1)^s \, r_{s-1} \sum_{j=1}^s (b_j-1-d_j) \prod_{i=1}^{s-1} c_i \quad (s \geq 2) \]

For instance,

\[ (-1)^s \, f_s = r_s \, L(x^s \, p_s) = r_s \, L[x^{s-1}(a_s p_{s+1} + b_s p_s + c_s p_{s-1})] = r_s \, c_s \, L[x^{s-1} p_{s-1}] \]
**Theorem:** Let $C$ be a code with distances $d_1, \ldots, d_s$. Let

$$D = b_0 + \cdots + b_{s-1} - d_1 - \cdots - d_s.$$  

(a) Suppose that $c_i < 0, i = 1, 2, \ldots$ and $D > 0$. Then

$$|C| \leq h_0 + h_1 + \cdots + h_{s-2} + h_s.$$  

(b) Suppose that $c_i > 0, i = 1, 2, \ldots$. Then

$$|C| \leq \begin{cases} 
  h_0 + h_1 + \cdots + h_{s-2} & s \equiv 1 \pmod{2} \text{ and } D > 0 \\
  h_0 + h_1 + \cdots + h_{s-1} & s \equiv 1 \pmod{2} \text{ and } D < 0 \\
  h_0 + h_1 + \cdots + h_{s-2} + h_s & s \equiv 0 \pmod{2} \text{ and } D < 0.
\end{cases}$$
Applications: Intersecting families

Family \( \mathcal{F} = \{F_1, F_2, \ldots \} \) of subsets of an \( n \)-element set is called

- \textit{\( w \)-uniform} if \( \forall_i |F_i| = w \)
- \textit{\( s \)-intersecting} if \( \forall F_i, F_j |F_i \cap F_j| \in \{w, t_1, \ldots, t_s\} \) for some \( t_1, \ldots, t_s, 0 < t_i < w \).

**Theorem 1** (Ray-Chaudhuri–Wilson, 1975)

Let \( \mathcal{F} \) be a \( w \)-uniform \( s \)-intersecting family. Then \( |\mathcal{F}| \leq \binom{n}{s} \).

The indicator vectors of a \( w \)-uniform \( s \)-intersecting family \( \mathcal{F} \) form an \( s \)-code \( C \) in \( J_{n,w} \).

Johnson space \( J_{n,w}=\{x \in \{0,1\}^n: wt(x)=w\} \)

\( \{p_i(x), i=0,1,\ldots,w\} \) – discrete dual Hahn polynomials

For instance, let \( n=13, w=5, t_1=3, t_2=4 \), then we obtain \( |\mathcal{F}| \leq 78 \)

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Delsarte LP bound: \(|C| \leq \max\{1+a_1+a_2+\ldots+a_s : a_i \geq 0, \sum_{i=1}^s a_i p_k(i) \geq -1, 0 \leq k \leq \text{diam}(X)\}\)

\[
|\mathcal{F}|=1+\max\{a+b, \quad aQ_4(1)+bQ_4(2) \geq -Q_4(0); \quad \text{aQ}_5(1)+bQ_5(2) \geq -Q_5(0); \quad a,b \geq 0 \}
\]

\( Q_1(1)=27/40 \quad Q_1(2)=7/20 \)

\( Q_2(1)=2/5 \quad Q_2(2)=1/28 \)

\( Q_3(1)=7/40 \quad Q_3(2)=-17/280 \)

\( Q_4(1)=0 \quad Q_4(2)=-1/28 \)

\( Q_5(1)=-1/8 \quad Q_5(2)=-1/28 \)

\( (1/28)b \leq 1 \)

\( (1/8)a+(1/28)b \leq 1 \)

\( a=16, \quad b=28 \)

\(|\mathcal{F}| \leq 45 \)
**Theorem:** Let $\mathcal{F}$ be a $w$-uniform $s$-intersecting family. Suppose that

$$p_1 + \cdots + p_s > ws - \sum_{i=0}^{s-1} \frac{(n + 2)w(n - w) - ni(n - i + 1)}{(n - 2i)(n - 2i + 2)},$$

then $|\mathcal{F}| \leq \binom{n}{s} - \binom{n}{s-1} + \binom{n}{s-2} = \binom{n}{s} \left(1 - \frac{s}{n-s+1} \frac{n-2s+3}{n-s+2}\right)$.

For instance, let $s=2$
Consider codes (intersecting family) with distances (intersections) 1,2 ($w-1,w-2$) between distinct codewords. Make a code as follows

```
11111100000000000
111111001100000000
```

| $|C|$ | $=\frac{1}{2}(n-w+2)(n-w+1)$ |
|-------|----------------------------------|
| 111111001100000000 | |

In many cases for $6 \leq n \leq 46$ we establish maximality of this code:

<table>
<thead>
<tr>
<th>$n,w$</th>
<th>6 3 10 3 12 5</th>
<th>........</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 3 10 4</td>
<td>........</td>
<td>35 3</td>
</tr>
<tr>
<td>8 3 11 3 18 3</td>
<td>35 4</td>
<td></td>
</tr>
<tr>
<td>9 3 11 4 18 4</td>
<td>35 5</td>
<td></td>
</tr>
<tr>
<td>9 4 12 3 18 5</td>
<td>35 6</td>
<td></td>
</tr>
<tr>
<td>10 3 12 4 18 6</td>
<td>etc.</td>
<td></td>
</tr>
</tbody>
</table>
Generally for 2-distance constant-weight codes (intersecting families) we obtain

**Proposition:** Suppose that \( d_1 + d_2 \leq \frac{2w(n-w) - n}{n-2} \) (\( p_1 + p_2 \geq \frac{2w - (2w(n-w)-n)/(n-2)}{2} \)). Then

\[ |C| \text{ or } |\mathcal{F}| \leq \frac{1}{2}(n-1)(n-2) \]

Similarly, for 3-intersecting \( w \)-uniform families the bounding condition on the distances is computed to be

\[ d_1 + d_2 + d_3 < \frac{3w(n-w) - 3n}{n-4} \]
Applications: Binary codes

Let $C \subseteq H_n = \{0,1\}^n$ be a binary code with distances $d_1, d_2, \ldots, d_s$

**Theorem:** Suppose that $d_1 + \cdots + d_s < sn/2$. Then

$$|C| \leq 1 + n + \binom{n}{2} + \cdots + \binom{n}{s-2} + \binom{n}{s}.$$ 

**Theorem:** (a) Let $C \subseteq H_n$ be a binary code in which the distances between distinct codewords are $d_1, d_2$. If $d_1 + d_2 < n$ then $|C| \leq M_2 := 1/2(n^2 - n + 2)$.

(b) If $6 \leq n \leq 46, 48 \leq n \leq 52$, $n = 54, 55, 57, 58, 60, 61, 62, 63, 64, 66, 67, 69, 72$, then the size of a maximal code with 2 distances equals $M_2$.

Code $C = \{00\ldots00,110\ldots0,0110\ldots0,\ldots,00\ldots011\}$ has $\{d_1, d_2\} = \{2, 4\}$; is optimal

To prove that even if $d_1 + d_2 \geq n$, no larger code exists for small $n$, compute the Delsarte LP bound.

$$|C| \leq \max\{1+a_1+a_2+\ldots+a_s : a_i \geq 0, \sum_{i=1}^s a_ip_k(i) \geq -1, 0 \leq k \leq n\}$$
**Theorem:** (a) Let $C \subset H_n$ be a binary code in which the distances between distinct codewords are $d_1, d_2, d_3$. If $d_1 + d_2 + d_3 < 3/2n$, then $|C| \leq M_3 := n + \binom{n}{3}$.

(b) If $8 \leq n \leq 22$ or $n = 24, 25$ then then the size of a maximal code with 3 distances equals $M_3$.

Code $C=\{\text{all wt 1; all wt. 3}\}$ has \{d_1,d_2,d_3\}=\{2,4,6\}; is optimal

To prove that even if $d_1+d_2,d_3\geq n$, no larger code exists for small $n$, compute the Delsarte LP bound

**Theorem:** (a) Let $C \subset H_n$ be a binary 4-distance code with distances $d_1, d_2, d_3, d_4$. If $d_1 + d_2 + d_3 + d_4 < 2n$, then $|C| \leq M_4 := 1 + \binom{n}{2} + \binom{n}{4}$.

(b) For $10 \leq n \leq 24$ the maximum size of a 4-distance binary code equals $M_4$. 
Applications: Spherical codes

Let

\[ M_s := \binom{n + s - 1}{s} + \binom{n + s - 2}{s - 1}. \]


**Theorem:** Let \( C \subseteq S^{n-1} \) be a set of points \( (x_1, x_2, \ldots, x_m) \) such that \( (x_i, x_j) \in \{t_1, t_2, \ldots, t_s\} \)

Then \( m \leq M_s \)

O.R. Musin (2008): For \( s=2, t_1 + t_2 > 0 \) \( m \leq \frac{1}{2}n(n+1) \)

More generally:

**Theorem:** Suppose that \( s \) is even and \( t_1 + t_2 + \cdots + t_s > 0 \), then

\[ |C| \leq M_{s-2} + \binom{n - 2 + s}{s} + \binom{n + s - 3}{s - 1}. \]