Number of errors correctable with codes on graphs

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joint work with

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Linear binary codes constructed from regular sparse graphs

(a) LDPC codes
(b) bipartite-graph codes
(c) hypergraph codes
LDPC codes

\[ H = \begin{array}{c|c}
  c_i \\
  \hline
  v_j 
\end{array} \]

\[ (x_1, x_2, \ldots, x_n) \in A \quad (x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \in B \]

repetition code  single parity check code

Generalization: Bipartite-graph codes (Tanner)

\[ x_i + x_j + x_k = 0 \]

A, B – linear codes of length n
**Bipartite-graph codes**

G = \((V_0, V_1, E)\) balanced, n-regular, bipartite

\(|V_0| = |V_1| = m, \ |E| = N = nm\)

\(A[n, R_0 n]\) \(B[n, R_1 n]\) binary linear codes

**C \([N, RN, D]\) = parallel concatenation of A and B:**

\(x \in C \iff \forall v \in V_0 \ x(v) \in A, \forall w \in V_1 \ x(w) \in B\)

\(R \geq R_0 + R_1 - 1\)
Decoding

In the analysis we assume that the local codes are the same: $A=B \ [n,k,d_0=2t+1]$
Decoding

Left decoding round $L(y)$: in parallel decode all subvectors of the $N$–vector $y$ that appear in the neighborhoods of the left vertices with the code $A$

Right decoding round $R(y)$: in parallel decode all subvectors of the $N$–vector $y$ that appear in the neighborhoods of the right vertices with the code $A$

Iterative procedure: $y$ received from the channel. Perform

$$... R (L (R (L (R (L(y)))))...) ...$$

Proceed until a fixed point is encountered, or stop after $O(\log n)$ iterations (Zémor 2001)
Why this works: Property \( P_1(\sigma) \). For any subset \( S \subset V_0 \) (\( V_1 \)), \(|S| = \sigma m\) the number of vertices in \( V_1 \) (\( V_0 \)) whose S-degree exceeds \( t \) is 
\[ \leq (\sigma - \varepsilon)m \]

This enables one to prove convergence of decoding.

In previous works this property was controlled by the spectral gap of the graph.

However this approach imposes a constraint on \( d_0 = d(A) \) of the form \( d_0 \geq \sqrt{n} \). Our purpose in this work is to lift this constraint.
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\[
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Replace graphs with spectral gap with random graphs.
It is known that code ensembles constructed using random regular graph have the same weight distribution as codes obtained from graphs with large spectral gap.
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Replace graphs with spectral gap with random graphs.
It is known that code ensembles constructed using random regular graphs have the same weight distribution as codes obtained from graphs with large spectral gap.

We estimate the guaranteed number of errors in ensembles of codes constructed on random regular bipartite graphs with local codes \( A[n,k,d_0] \) for any given \( d_0 \).

The estimate is proved by estimating maximum \( \sigma \) for which the ensemble of graphs contains graphs that have Property \( P_1(\sigma) \).
**Theorem:** Local code $A[n, R_0, d_0=2t+1]$. The proportion of correctable errors is at least $\sigma t/n$, as defined below

\[ F_{n,t}(\sigma) = (n - 1) h(\sigma) \]

\[ F_{n,t}(\sigma) = \max_{z \in \mathcal{M}(t, \sigma)} \left( h(z) + \sum_{i=1}^{n} z_i \log \left( \binom{n}{i} \right) \right) \]

where

\[ \mathcal{M}(t, \sigma) = \left\{ z \in \mathcal{Z}_n : \sum_{i=1}^{n} iz_i = \sigma n, \sum_{i=t+1}^{n} z_i = \sigma \right\}, \]

\[ \mathcal{Z}_n = \{ z \in [0, 1]^{n+1} : \sum_{i=0}^{n} z_i = 1 \} \]

**Example:** $A[23,12,7]$ binary Golay code. Proportion of errors 0.00063.
What happens when \( n \) becomes large, \( t = \tau n \)?

Ensemble \( \mathcal{G}(A, m) \) contains codes that correct \( \sigma \tau N \) errors for any \( \sigma \leq \sigma^* \), where \( \sigma^* \) is given by

\[
\sigma^* = \sup \left\{ \sigma > 0 : \forall 0 < x < \sigma \ (1-x)h\left(\frac{x(1-\tau)}{1-x}\right) + xh(\tau) + \varepsilon_n < h(x) \right\}
\]

If \( n \to \infty \), the proportion of correctable errors approaches \( \tau^2 \)
Another view of bipartite-graph codes:

instead of

consider the edge-vertex graph of \( G \):
Each variable node is connected to $\geq 3$ different check vertices.

This ensemble was introduced by Gallager; generalized by Bilu and Hoory '04. under the name “hypergraph codes”
Hypergraph codes

$l$-partite, $n$-regular, uniform hypergraph $H=(\{V_1,V_2,\ldots,V_l\},E)$; $|V_1|=\ldots=|V_l|=m$

Bits of the codeword correspond to (hyper)edges
Every vertex imposes a set of local constraints on the bits adjacent to it
$A[n,k=R_0n,d_0]$ linear local code

$$R(C) \geq lR_0-(l-1)$$


Here we consider the ensemble of codes $H(A,l,m)$ defined by random regular (hyper)graphs and a local code $A[n,R_0n,d_0]$.

Using the above algorithm and probabilistic analysis, we estimate the proportion of errors correctable with codes from the ensemble $H(A,l,m)$ as $\gamma_{mn}$, where

$$
\gamma^* = \sup \{ x > 0 : \forall 0 < \gamma \leq x (l/n)F_{n,i}(\gamma) < (l-1)h(\gamma) \}.
$$

$$
M(t, \gamma) = \left\{ z \in \mathbb{Z}_n : \sum_{i=1}^{n} iz_i = \gamma n, \sum_{i=1}^{t} iz_i = \sum_{i=d_0-t}^{n} tz_i \right\}.
$$

We compute examples with local Hamming codes.

We also estimate the proportion of correctable errors for large length of the local code $A$. 