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Bounds on the Covering Radius of Linear Codes*

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Abstract. Asymptotically bounding the covering radius in terms of the dual distance is a well-studied problem. We will combine the polynomial approach with estimates of the distance distribution of codes to derive new results for linear codes.

Keywords: asymptotics of Krawtchouk polynomials, Christoffel-Darboux kernel, distance distribution, dual distance

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1. Introduction

Let $F = \mathbb{F}_2^n$ and let $C \subset F$ be a binary linear code of length n. The covering radius of C is defined as

$$r(C) = \max_{x \in F} \min_{y \in C+x} \operatorname{wt}(x),$$

where $\operatorname{wt}(\cdot)$ denotes the Hamming weight. Bounding the covering radius of codes is one of the main extremal problems of coding theory. Let $C' = \{x \in F : (x, c) = 0 \ \forall c \in C\}$ be the dual code of C and $(A'_i, 0 \le i \le n)$ be its weight distribution, where $A'_i = |\{x \in C' : \operatorname{wt}(x) = i\}|$ and (x, c) denotes the dot product. The minimal $i \ge 1$ such that $A'_i > 0$ is called the dual distance of C, denoted by d'.

The most developed direction is deriving bounds on r in terms of the strength of C as a design in F or, in other words, in terms of the dual distance d'. This problem received considerable attention through the last decade, see [2], [3, Ch. 12], [4,6–8,11–19], and this is the problem studied in the present paper. Let $r(d') = \max r(C)$ where the maximum is taken over all codes of dual distance at least d'. We will be interested in asymptotic upper bounds on $\rho = r/n$ valid for any sequence of codes C_n of growing length n and dual distance $d' \ge \delta' n$. A review of the methods used for obtaining such bounds is given

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in [2,7], see also [11]. Previous work has been largely concentrated around an application of Delsarte's polynomial method put forward by Tietäväinen in [19]. The result in that paper has the following form:

$$\rho(\delta') \le \varphi(\delta'/2), \tag{1}$$

where $\varphi(x) = \frac{1}{2} - \sqrt{x(1-x)}$. Ultimately the best upper bounds on $\rho(\delta')$ known are obtained based on the following theorem. By $K_i(x)$ we denote the Krawtchouk polynomial of degree i.

THEOREM 1 [3, p. 230]. *Let s be an integer, and let $f(x) = \sum_{i=0}^{n} f_i K_i(x)$ be a polynomial such that $f(i) \le 0$, $i = s+1, \ldots, n$, and*

$$f_0 > \sum_{j=d'}^{n} |f_j| A'_j \tag{2}$$

Then the covering radius of a linear code C with dual distance d' is at most s.

So far applications of this theorem were based on bounding A'_j above by the maximal size $A(n, d', j)$ of a code of length n, distance d' and constant weight j. This enables one to derive an upper bound of the form $\sum_{j=d'}^{n} |f_j| A'_j \le F$; then to establish (2) for a given polynomial $f(x)$ one only needs to verify that $F < f_0$. Asymptotically this approach amounts to using bounds on the rate $R(\delta', \xi)$ of constant-weight codes with relative distance δ' and relative weight $\xi = j/n$. In particular [2] relies upon the "JPL" bound [13], and [7] uses an improvement of this bound from [9,14] to improve the bound on $\rho(\delta')$ for $0.04 \le \delta' \le 0.20$. Papers [2,7] use the polynomial

$$W_t(x) := \frac{(K_{t+1}(x) + K_t(x))^2}{a - x}, \tag{3}$$

where a is the smallest root of the numerator and t an appropriately chosen parameter. This polynomial was first suggested in [13] for bounding the size of codes. Jointly papers [2] and [7] together with [6] contain the best bounds known to-date. The first two bounds are cited in Theorem 4 below and the remarks following its proof; the third one is too cumbersome to reproduce here (see also Theorem 2 in [2]).

In this paper we suggest to replace estimates of $R(\delta', \xi)$ with other bounds on the weight distribution of codes. One option is, for a given choice of the polynomial $f(x)$, to bound the sum $\sum_{j=d'}^{n} |f_j| A'_j$ as a whole. A method for deriving universal bounds of this type was suggested in a recent work [1]. In the same paper we also obtained estimates of individual coefficients A_i which can be used in Theorem 1. Both approaches enable us to improve the cited results. We establish the following new bounds (logarithms are base 2 throughout).

THEOREM 2. *For $0 \le \delta' \le 1/2$*

$$\rho(\delta') \le \varphi(H^{-1}(1 - H(\varphi(\delta')))) \tag{4}$$

$$\rho(\delta') \le 2\varphi(\delta'), \tag{5}$$

where $H(x) = -x \log(x) - (1-x)\log(1-x)$.

Table 1. Bounds on $\rho(\delta')$.

δ'	0.28	0.32	0.36	0.4	0.44	0.48
(4)	0.1047	0.0754	0.0499	0.0285	0.0120	0.00177
(5)	0.102	0.067	0.04	0.020	0.0072	0.00080
[2]	0.105	0.0769	0.051	0.029	0.0125	0.00178
[6]	0.149	0.096	0.057	0.029	0.0103	0.0011

Both (4) and (5) improve the known results (see Table 1). In particular the bound of [6] is improved for all δ' and the bound of [2] in the interval $\delta' \in [\delta'_1, 1/2)$, where $\delta'_1 \approx 0.27$. For $\delta' \le 0.267\ldots$, (4) is better than (5). For $0 < \delta' \le \delta'_1$ the results of [2,7] remain the best known. As shown in Table, the improvement of the previous results given by Theorem 2 is rather substantial.

Apart from proving Theorem 2 we also present a new proof of the results of [2,7] which enables us to formulate them in a more explicit manner than in the original works.

2. Bounds on the Weight Distribution

In this section we collect the results from [2] that we need. We also prove an extended version of one of the estimates on the weight distribution from [2].

A. Krawtchouk Polynomials (for the proofs see, for instance, [3]). Let $\alpha(i) = 2^{-n}\binom{n}{i}$. We have $\sum_{i=0}^{n} K_t(i)K_s(i)\alpha(i) = \binom{n}{t}\delta_{t,s}$ and thus $\|K_t\|^2 := \sum_{i=0}^{n}(K_t(i))^2\alpha(i) = \binom{n}{t}$. This implies that

$$(K_t(i))^2 \le \binom{n}{t}2^n \Big/ \binom{n}{i}. \tag{6}$$

The Krawtchouk coefficients of any polynomial $f(x) = \sum f_j K_j(x)$ can be computed from $\|K_j\|^2 f_j = \sum_{i=0}^{n} f(i)K_j(i)\alpha(i)$. In particular,

$$K_a(x)K_b(x) = \sum_{c=0}^{n} p_{a,b}^{c} K_c(x), \tag{7}$$

where

$$p_{a,b}^{c} = \binom{c}{(b-a+c)/2}\binom{n-c}{(b+a-c)/2}\chi\{a+b-c \in 2\mathbb{Z}\} \tag{8}$$

Note that $p_{a,b}^{c} = 0$ for $a+b < c$ and $p_{a,b}^{0} = \delta_{a,b}\binom{n}{a}$.

With $t = \tau n$ and $i = \mu n$ we see from (6) that $n^{-1}\log|K_t(i)| \le E_1(\tau, \mu) + o(1)$, where

$$E_1(u, v) = \frac{1}{2}(H(u) - H(v) + 1).$$

Let $\tau < 1/2$. The zeros of $K_t(x)$ are located inside the segment $[n\varphi(\tau), n(1-\varphi(\tau))]$ and for the minimum zero we have

$$n\varphi(t/n) \le x_t \le n\varphi(t/n) + t^{1/6}\sqrt{n-t}. \tag{9}$$

Note that the function $\varphi(x)$ is monotone decreasing for $0 \le x \le 1/2$. It is also an involution and so $\varphi^2 = \text{id}$. Let

$$I(\tau, \mu) = \int_0^\mu \log \frac{s + \sqrt{s^2 - 4y(1-y)}}{2-2y}\, dy$$

where $s = 1 - 2\tau$. It is known [5] that $n^{-1} \log K_{\lfloor \tau n \rfloor}(\mu n) = E_2(\tau, \mu) + o(1)$, where

$$E_2(\tau, \mu) := H(\tau) + I(\tau, \mu) \quad (0 \le \mu \le \varphi(\tau)). \tag{10}$$

An upper bound on the exponent of $K_t(i)$ which has a simpler form and is not as crude as E_1 was derived in [10]. It has the form $n^{-1} \log K_{\lfloor \tau n \rfloor}(\mu n) \le E_3(\tau, \mu) + o(1)$, where

$$E_3(\tau, \mu) := \frac{1}{2}\left[\mu \log \frac{\varphi(\tau)}{1-\varphi(\tau)} + \log(1-\varphi(\tau)) + H(\tau) + 1\right] \quad (0 \le \mu \le \varphi(\tau)). \tag{11}$$

We note that $E_1(\tau, \mu) \ge E_3(\tau, \mu) \ge E_2(\tau, \mu)$ for $0 \le \mu \le \varphi(\tau)$ with equality if and only if $\mu = \varphi(\tau)$. Moreover, for this μ also

$$(E_1(\tau, \mu))'_\mu = (E_2(\tau, \mu))'_\mu = (E_3(\tau, \mu))'_\mu.$$

B. Weight Distribution. Let $a_\xi(C) = (\log A_{\xi n})/n$ be the exponent of the weight coefficient of a linear code C of length n. We denote

$$a_\xi(\delta) = \max_{C:\, d(C)=\delta n} a_\xi(C).$$

THEOREM 3. *For any sequence of codes of relative distance δ*

$$a_\xi \lesssim \begin{cases} H(\xi) + H(\varphi(\delta)) - 1 & \delta \le \xi \le 1-\delta \\ -2I(\varphi(\delta), \xi) & 1-\delta \le \xi \le 1. \end{cases} \tag{12}$$

(See Figure 1).

Proof. (outline).

(a) The starting point is the following result of [1]. Let $g(x)$ be a function and $Z(x)$ a polynomial over $\mathbb{R}$. Suppose that $Z(x) = \sum_i z_i K_i(x)$ and that $z_i \le 0$, $0 \le i \le n$. If $Z(i) \ge g(i)$, $d \le i \le n$ then

$$\sum_{i=d}^{n} g(i) A_i \le z_0 |C| - Z(0). \tag{13}$$

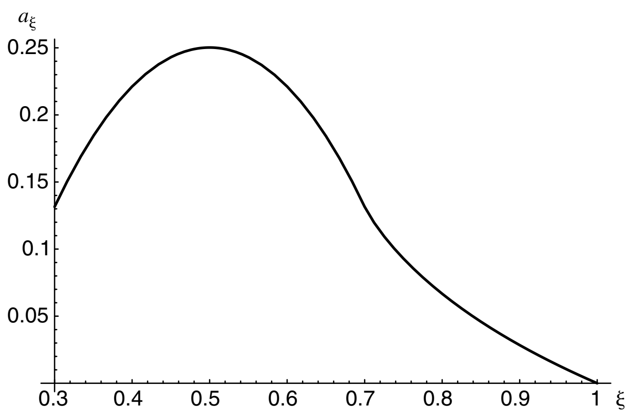

Figure 1. Upper estimate of the exponent a_ξ of the distance spectrum for a family of codes with distance $\delta = 0.3$.

(b) Take $\tau = \varphi(\delta)$ and $t = \tau n$. As in [1], we take

$$g(i) = (K_w(i))^2,$$
$$Z(i) = (K_w(i))^2 - \frac{t+1}{2}\frac{\binom{n}{w}}{\binom{n}{t}} W_t(i),$$

for a suitably chosen w. It is proved in [1], Prop. 2 that for large n and $\frac{w}{n} = \omega < \frac{t}{n} = \tau$, this choice satisfies the conditions of (a). Computing the right-hand side of (13) we get

$$\sum_{i=d}^{n} (K_w(i))^2 A_i \le \frac{n^2(t+1)}{2at^2}\binom{n}{w}\binom{n}{t} - \binom{n}{w}^2. \tag{14}$$

(c) Let $\delta \le \xi \le 1/2$ and take $\omega = \varphi(\xi)$. Then we have $\omega < \tau = \varphi(\delta)$. Taking logarithms in (14) and bounding the exponent of $K_w(i)$ by $E_1(\omega, \xi)$ we obtain the first inequality in (12).

(d) Let $1/2 \le \xi \le 1-\delta$. Then we take the same polynomials $g(i)$ and $Z(i)$ with $\tau = \varphi(\delta)$ but choose $\omega = 1 - \varphi(\xi)$. The argument of (c) is then repeated which is possible since $|K_k(x)| = |K_k(n-x)|$.

(e) Now let $1-\delta \le \xi \le 1$. Since we want to ensure that $\omega < \tau$, we replace the above choice of ω by a number arbitrarily close to τ. Then a better estimate of $K_w(i)$ is given in (10), and we get the second inequality of the statement. ∎

Note that (14) implies the following estimate of the left-hand side: for any sequence of codes with distance d

$$n^{-1} \log \sum_{i=d}^{n} (K_w(i))^2 A_i \lesssim H(\omega) + H(\tau) \quad (\omega \le \tau). \tag{15}$$

One more result from [1] that we use is as follows:

$$n^{-1}\log\sum_{i=d}^{n} p_{t,t}^{i} A_i \lesssim 2H(\tau)-H(\delta/2) \quad (\delta/2\le\tau). \tag{16}$$

3. Bounds on the Covering Radius

We will use the estimates of the previous section for the weight distribution $(A'_i, d'\le i\le n)$ of the code C'. By abuse of notation we let $a_\xi = n^{-1}\log A'_i$ and denote by $a_\xi(\delta')$ a generic upper bound on a_ξ that holds for any family of codes with relative distance δ'. The following theorem gives results of [2,7] in an analytic form, replacing a numerical procedure employed there.

THEOREM 4. *Let τ be the minimal number such that*

$$\max_{\delta'\le\xi\le 2\tau}\left\{(1-\xi)H\left(\frac{\tau-\xi/2}{1-\xi}\right)-H(\tau)+\xi+a_\xi(\delta')\right\}<0, \tag{17}$$

Then $\rho\le\varphi(\tau)$.

This theorem will follow immediately if we establish the asymptotic behavior of the Krawtchouk coefficients of $f(x)$ from (3).

LEMMA 5. *Let $f(x)$ be the polynomial (3) and let $f(x)=\sum_{j=0}^{n} f_j K_j(x)$ be its Krawtchouk expansion. Then*

$$\log f_j \sim \log p_{t,t}^{j}. \tag{18}$$

Suppose that $j=\xi n$ and $t=\tau n$. Then

$$n^{-1}\log f_j = (1-\xi)H\left(\frac{\tau-\xi/2}{1-\xi}\right)+\xi+o(1)$$

Proof. The expression for f_j has the form [13]:

$$f_j = \frac{2}{(t+1)K_t(a)}\binom{n}{t}\sum_{i=0}^{t}\frac{K_i(a)}{\binom{n}{i}}\left(p_{t,i}^{j}+p_{t+1,i}^{j}\right), \tag{19}$$

and $f_j\ge 0$ for all $0\le j\le n$. Note that for any i if $p_{t,i}^{j}\ne 0$ then $p_{t,i+1}^{j}=0$ and vice versa. Estimating (19) below by the last term, we get

$$f_j \ge \frac{2}{t+1}p_{t,t}^{j}(1+O(n))$$

which implies the expression of the Lemma as a lower bound. Let us prove that this is also an upper bound. Since[1]

$$\binom{n}{a}K_t(a)=\binom{n}{t}K_a(t)$$

we can rewrite (19) as

$$f_j = \frac{2}{(t+1)K_t(a)} \frac{\binom{n}{t}}{\binom{n}{a}} \sum_{i=0}^{t} K_a(i)\big(p^j_{t,i} + p^j_{t+1,i}\big). \tag{20}$$

Note that the main term of the minimal zero x_a of $K_a(x)$ behaves as $n\varphi(a/n)$, and by definition $a = n\varphi(t) + o(n)$. Hence $x_a \sim t$, and so the sum on i ranges over the segment $0 < i < x_a$. Now let us estimate $K_a(i)$ from above by $\exp(nE_3(\varphi(\tau), \xi))$ (see (11)) and take logarithms. Putting $i = \mu n$ and recalling that φ is an involution we can bound the exponent of the summation term in (20) as follows:

$$\frac{1}{2}\Big[\mu \log \frac{\tau}{1-\tau} + \log(1-\tau) + H(\varphi(\tau)) + 1\Big] + \xi H\left(\frac{\tau - \mu + \xi}{2\xi}\right) + (1-\xi)H\left(\frac{\tau + \mu - \xi}{2(1-\xi)}\right). \tag{21}$$

The derivative of this expression on μ has the form

$$M(\mu) = \log \frac{(\xi + \tau - \mu)(2 - \xi - \tau - \mu)\tau}{(\xi - \tau + \mu)(\tau + \mu - \xi)(1-\tau)}.$$

Note that $0 \le \mu \le \tau \le 1/2$. We easily check that $M(\tau) \ge 0$ for any $0 \le \xi \le 2\tau$. Next we prove that $M(\mu)$ has no zeros for $\tau - \xi \le \mu \le \tau$. If it does, then these zeros, which satisfy the expression

$$\frac{(\xi + \tau - \mu)(2 - \xi - \tau - \mu)\tau}{(\xi - \tau + \mu)(\tau + \mu - \xi)(1-\tau)} = 1,$$

and therefore are equal to

$$\mu_{1,2} = \frac{\pm\sqrt{\xi^2 - 4(1-\tau)\tau(\xi^2 - \tau(1-\tau))} - \tau}{1 - 2\tau},$$

fall in the segment $\tau - \xi \le \mu \le \tau$. However, $\mu_2 < 0$, and it is checked directly that $\mu_1 > \tau$. Hence the term in (21) attains its maximum on μ for $\mu = \tau$ and we finally obtain

$$n^{-1} \log f_j \lesssim E_3(\varphi(\tau), \tau) + n^{-1} \log p^j_{t,t}.$$

Finally since $E_3(\tau, \varphi(\tau)) = E_1(\tau, \varphi(\tau))$, we can substitute E_1 together with the logarithm of $p^j_{t,t}$. This implies that the expression in the statement of the lemma is also an upper bound on the exponent of f_j and completes the proof. ■

Proof of Theorem 4. For $j = \xi n$ we bound above the exponent of A'_j by $a_\xi(\delta')$. Asymptotically the sum $\sum_j f_j A'_j$ is dominated by the largest term, j_0 say, such that its exponent attains the maximum on $\xi = j/n$ for $\delta' \le \xi \le 2\tau$. It is also immediate from Lemma 5 that

$$\log f_0 = n(H(\tau) + 2E_1(\tau, \varphi(\tau))) + o(n).$$

Computing $\log f_0 - \log(f_{j_0} A_{j_0})$, we obtain the expression under the maximum in (17). Together with Theorem 1 this establishes the claim. ■

Obtaining bounds on $\rho(\delta')$ with this theorem is a matter of choosing a suitable bound on $a_\xi(\delta')$. An obvious idea is to use upper bounds on constant weight codes:

$$a_\xi(\delta') \le R(\delta', \xi).$$

Substituting the bound on $R(\delta', \xi)$ from [13] we obtain the result of [2].[2] An improved bound from [14] gives the result of [7].

A better result for large δ' is obtained if we combine Theorem 4 with Theorem 3. The result can be expressed in a closed form. We need to substitute the bound (12) into (17) and optimize on ξ, $\delta' \le \xi \le 1$. The expression whose maximum on ξ is sought, is different for $\xi \le 1-\delta'$ and $\xi > 1-\delta'$. However in both cases this maximum is attained for $\delta' \le \xi \le 1-\delta'$. Indeed, suppose that $1-\delta' \le \xi \le 1$, then substituting the second inequality in (12) into (17), we observe that the part of the expression that depends on ξ equals $(n^{-1}\log p_{t,t}^{\xi n}) - 2I(\varphi(\delta), \xi)$. The first logarithm is the falling function of ξ since its derivative equals

$$\frac{d}{d\xi}\left((1-\xi)H\left(\frac{\tau-\xi/2}{1-\xi}\right)+\xi\right) = \frac{1}{2}\log\frac{(2\tau-\xi)(2-2\tau-\xi)}{(1-\xi)^2},$$

which has no zeros and is negative for $0 \le \xi \le 2\tau$. The function $-2I(\varphi(\delta'), \xi)$ is also directly checked to be falling on ξ. Thus the maximum in this case is attained for $\xi = 1-\delta'$ but for this ξ the bound is the same as in the first case of (12). It remains to analyze the case $\delta' \le \xi \le 1-\delta'$. Substituting the first upper bound (12) into the expression to be maximized in (17), we obtain

$$(1-\xi)H\left(\frac{\tau-\xi/2}{1-\xi}\right) - H(\tau) + \xi + H(\xi) - H(\varphi(\delta)) + 1.$$

This function has a unique maximum on ξ for $\xi = 2\tau(1-\tau)$. Substituting this value of ξ, we get, upon simplification, the expression

$$H(\tau) - 1 + H(\varphi(\delta)).$$

The minimum τ for which this is negative is thus arbitrarily close to $\tau_0 := H^{-1}(1 - H(\varphi(\delta)))$. Thus $\rho(\delta') \le \varphi(\tau_0)$, which proves the first part of Theorem 2.

To prove the second part of this theorem, let us take in Theorem 1 the polynomial $f(x)$ given by $f(i) = 2^n p_{t,t}^i$, where $t/n = \tau = \varphi(\delta')$. By (7)–(8) we get $f(i) = 0\,(2t+1 \le i \le n)$ and $f_i = (K_t(i))^2$. Now from (15) we have

$$n^{-1}\log\sum_{i=d'}^{n}(K_t(i))^2 A_i \le 2H(\tau)(1+o(1)).$$

Since $n^{-1}\log f_0 \sim 2H(\tau)$, this choice of $f(x)$ satisfies for large n the conditions of Theorem 1. So $\rho \le 2\tau = 2\varphi(\delta')$, as was to be proved.

Remark. If we take $f(x) = W_t(x)$ then by (7) $\log f_i \sim \log p_{t,t}^i$, and so

$$f_0 \sim \binom{n}{t} = \exp(nH(\tau) + o(n)).$$

Hence together with (16) we see that (2) is satisfied if τ is arbitrarily close to but less than $\delta'/2$. The first zero of $f(x)$ behaves as $n\varphi(\delta'/2)$ and for greater x, $f(x)$ stays nonpositive; hence $\varphi(\delta'/2)$ is an upper bound on ρ. This gives another proof of (1).

Notes

1. Strictly speaking, for this to be true, a must be an integer. However since our aim is asymptotic results, the effect of rounding is insignificant.
2. In fact, [2] does a little more: the authors there substitute the bound from [13] and some recurrence relations on the function $R(\delta', \xi)$ to improve the result for small δ'.

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