
1. We have proved (Thm. 2.6) that there exists an \([n,k]\) binary linear code whose weight distribution \(A_1, A_2, \ldots, A_n\) is bounded above as
\[ A_w \leq n^2 \binom{n}{w} 2^{k-n} \]
if the right-hand side is at least one (and \(A_w = 0\) otherwise). Prove that it is possible to improve this estimate by replacing \(n^2\) with \(n\): there exists a binary linear \([n,k]\) code with
\[ A_w \leq n \binom{n}{w} 2^{k-n}. \]
Hint: consider the ensemble of all linear \([n,k]\) codes. Find the probability that a nonzero vector is contained in a random code from the ensemble. Compute the expected number of weight-\(w\) vectors, use the Markov inequality.

2. Consider a finite metric space \(X\) in which the volume of the ball \(B_r(x)\) depends on its center. Let \(\langle B_r \rangle = \frac{1}{|X|} \sum_{x \in X} \text{vol}(B_r(x))\) be the average volume of the ball of radius \(r\). Prove that \(X\) contains a code of minimum distance \(d\) and size \(M \geq \frac{|X|}{4 \langle B_{d-1} \rangle} \).
Hint: perform the Gilbert procedure on the subset of all the points \(y \in X\) such that \(\text{vol}(B_{d-1}(y)) \leq 2 \langle B_r \rangle\).

3. (GV bound in the Johnson space) Let \(\mathcal{J}^{n,w}\) be the space of all binary vectors of length \(n\) and weight \(w\).
(a) Let \(x, y \in \mathcal{J}^{n,w}\). Prove that the Hamming distance \(d(x, y)\) is even.
(b) Let \(C\) be a code of rate \(R\) and minimum distance \(d = 2\delta n\). Prove that \(\mathcal{J}^{n,w}\) contains codes of rate approaching the bound
\[ R = h_2(\omega) - \omega h_2\left(\frac{\delta}{\omega}\right) - (1 - \omega) h_2\left(\frac{\delta}{1 - \omega}\right). \]

4. Recall the inequality used to prove Thm. 5.1
\[ P_e(C) \leq \min_{t} \left[ P_e(C, y \in B_t(0)) + P(y \notin B_t(0)) \right]. \]
where \(y\) is the the error vector. In the proof we took \(C\) a random linear code and chose \(t = d_{GV}\). Prove that for large \(n\) this choice is optimal, i.e., furnishes the minimum on \(t\) in (1).
Hint: Find the smallest \(t\) satisfying, for large \(n\),
\[ 2^{n(1-R)} \sum_{w=d}^{2t} \sum_{r=[w/2]}^{t} \sum_{i=[w/2]}^{w} \binom{n}{w} \binom{w}{i} (n-w)^r \left(\frac{r}{w-i}\right) (1-p)^{n-r} \geq \sum_{r=t+1}^{n} \binom{n}{r} p^r (1-p)^{n-r}. \]

5. Consider an \([n=8,k,d]\) linear code \(C\) given by the row space of the matrix
\[
G = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\end{pmatrix}
\]
Find \(k,d\). Write out a parity-check matrix of \(C\). Let \(E = \{1,2,3\}\). Write out the parameters of the codes \(C_E, C^E, (C^\perp)_E, (C^\perp)^E\). Explain all answers.

6. Let \(C[n,k,d]\) be a linear \(q\)-ary MDS code, i.e., a code with \(d = n - k + 1\). Prove that the dual code \(C^\perp\) is also MDS.