Homework 4

Please note that for a random variable \( X \), \( H(X) \) is the entropy. And, if no confusion arises, we denote \( H(a) \triangleq -a \log a - (1 - a) \log(1 - a) \) for a scale value \( a \in [0, 1] \).

**Problem 7.2**

(a) If \( a \neq \pm 1 \), it is clear that \( H(X|Y) = 0 \), i.e., based on the observation of \( Y \), \( X \) is distinguishable. So,

\[
I(X;Y) = H(X) - H(X|Y) = H(X) \leq 1 \tag{1}
\]

where the upper bound 1 is the capacity of this channel, which is achieved by uniform distribution.

(b) If \( a = \pm 1 \), it turns out to be a binary erasure channel with the capacity

\[
C = 1 - p = 1 - \frac{1}{2} = \frac{1}{2} \tag{2}
\]

**Problem 7.8**

Denote \( P(X = 0) \) as \( p \). Then

\[
H(Y|X) = \sum_{x \in \{0,1\}} H(Y|X = x) = p \cdot 0 + (1 - p) \cdot 1 = 1 - p \tag{3}
\]

and

\[
H(Y) = H \left( p + \frac{1}{2}(1 - p), \frac{1}{2}(1 - p) \right) = H \left( \frac{1}{2}(1 + p), \frac{1}{2}(1 - p) \right) \tag{4}
\]

Therefore

\[
I(X;Y) = H(Y) - H(Y|X) = -\frac{1}{2}(1 + p) \log \left[ \frac{1}{2}(1 + p) \right] - \frac{1}{2}(1 - p) \log \left[ \frac{1}{2}(1 - p) \right] - (1 - p) \tag{5}
\]

Take the derivative and set equal to zero

\[
0 = \frac{d}{dp} I(X;Y) = -\frac{1}{2} \log(1 + p) + \frac{1}{2} \log(1 - p) + 1 \tag{6}
\]

i.e.,

\[
\log \frac{1 + p}{1 - p} = 2 \tag{7}
\]

which yields the optimal \( p^* \) as \( p^* = \frac{3}{5} \). So the capacity of this Z-channel is

\[
C = H \left( \frac{1}{5} \right) - \frac{2}{5} \approx 0.322 \tag{8}
\]

Problem 7.23

(a) Denote \( P(X = 1) \) as \( p \). Then \( P(Y = 1) = P(XZ = 1) = P(X = 1, Z = 1) = p \alpha \). Therefore

\[
I(X;Y) = H(Y) - H(Y|X) = H(p \alpha) - pH(\alpha) - (1 - p) \cdot 0 = H(p \alpha) - pH(\alpha) \tag{9}
\]


Take the derivative and set equal to zero
\[ 0 = \frac{d}{dp} I(X;Y) = \alpha \log \frac{1 - p\alpha}{p\alpha} - H(\alpha) \] (10)
i.e., the optimal distribution \( p^* \) is
\[ p^* = \frac{1}{\alpha + \alpha \cdot 2^{H(\alpha)/2}} \] (11)
Therefore
\[ C = \max_p I(X;Y) = H(p^*) - p^* H(\alpha) \] (12)

(b) Note that
\[ I(X;Y,Z) = H(Y,Z) - H(Y,Z|X) \] (13)
\[ = H(Z) + H(Y|Z) - H(Z|X) - H(Y|Z,X) \] (14)
\[ = H(Y|Z) \] (15)
\[ = \alpha H(p) \] (16)
\[ \leq \alpha \] (17)
where Eqn.(15) is due to \( H(Z|X) = H(Z) \) and \( H(Y|Z,X) = 0 \). So the capacity is \( \alpha \) which is achieved by uniform distribution, i.e., \( p^* = \frac{1}{2} \).

Problem 7.34
There are two methods to solve this problem. Either use the result of Problem 7.28, or write down the mutual information explicitly and find the optimal distribution.

(a)
\[ C = \log (2^{C_1} + 2^{C_2}) \] (18)
where \( C_1 = C_2 = 1 - H(p) \). So
\[ C = \log \left[ 2 \times 2^{1-H(p)} \right] = 2 - H(p) \] (19)

(b) Again, we have
\[ C = \log (2^{C_1} + 2^{C_2}) \] (20)
where \( C_1 = 1 - H(p) \) and \( C_2 = 0 \). So
\[ C = \log \left[ 2^{1-H(p)} + 1 \right] \] (21)
Comment: It is strongly recommended to solve Problem 7.28 first. If you do not use the result of Problem 7.28, you will have the following (similar) result
\[ C = H(\alpha) + \alpha \left[ 1 - H(p) \right] \] (22)
where
\[ \alpha = \frac{1}{1 + 2^{H(p)-1}} \] (23)
It is easy to check that Eqn.(21) is equivalent to Eqn.(22).

Problem 7.28 appeared as Prol. 3. in Exam 1 whose solutions are posted on the class page.