Problem 1.

(a) Since \( C(\beta) = \max_{\alpha \leq \beta} I(\alpha) \), and \( EX = P_X(1) \), we can write
\[
P_X(1) \leq \beta \quad \text{we can write}
\]
\[
C(\beta) = \max_{\alpha \leq \beta} I(\alpha)
\]
\[
(b) \text{First let us show that } \frac{f(\beta)}{\beta} \downarrow \text{ for } \beta \in (0, \infty). \text{ Indeed, let } 0 < \beta_1 < \beta_2

\text{and write } \beta \text{ as a convex combination of the two ends of the segment :}

\[
\beta = (1-\lambda) \beta_1 + \lambda \beta_2, \text{ where } \lambda = \frac{\beta_1}{\beta_2}
\]

By concavity of \( f \) we have
\[
f(\beta) \geq (1-\lambda)f(\beta_1) + \lambda f(\beta_2) = \frac{\beta_1}{\beta_2} f(\beta_1) + \frac{\beta_2}{\beta_2} f(\beta_2) \Rightarrow \frac{f(\beta)}{\beta} \geq \frac{f(\beta_1)}{\beta_1}
\]

We know that \( I(\beta) \) is a concave function of \( \beta \), so let \( \beta_0 \) be a
unique maximizer of \( I(\beta) \). Since \( I(\beta) \) grows on \([0, \beta_0]\) and decreases on \([\beta_0, 1]\),
we can write
\[
(*) \quad C(\beta) = \begin{cases} I(\beta) & \text{if } \beta \leq \beta_0 \\ I(\beta_0) & \text{if } \beta > \beta_0 \end{cases}
\]

Continue with the solution of (b):

We know that \( \frac{C(\beta)}{\beta} \) is a decreasing function of \( \beta \in [0, 1] \). At the same time,
\[
\frac{C(\beta)}{\beta} \equiv \begin{cases} I(\beta)/\beta & \beta \leq \beta_0 \\ I(\beta_0)/\beta & \beta > \beta_0 \end{cases}
\]

Thus, \( \sup_{\beta > \beta_0} \frac{C(\beta)}{\beta} \) is achieved for \( \beta \to 0 \). 