ENEE627: Some problems on channel capacity and related questions.

The below problems serve the purpose of improving your understanding of the topic of channel capacity.

1. Let $M_1 = \{1, \ldots, M_1\}$ and $M_2 = \{1, \ldots, M_2\}$ be two finite sets (of messages) and let $g : M_1 \times M_2 \to X^n$ be an encoder that maps a pair $(m_1, m_2)$ to a codeword $x^n \in \{0, 1\}^n$.

Let $W_i, i = 1, 2$ be a message chosen from $M_i$ with uniform distribution. Suppose that the codeword $X^n = g(W_1, W_2)$ is transmitted over two independent binary erasure channels, namely BEC($q$) and BEC($p$), where $q > p$. Let $Y^n$ be the output of BEC($p$) and let $Z^n$ be the output of BEC($q$).

(a) Show that $I(W_1, W_2; Z^n) \leq n(1 - q)$.

(b) Show that $I(W_1; Z^n, W_2) \leq n(1 - q)$.

(c) Show that $I(W_1; Z^n) = I(W_1, W_2; Z^n) - I(W_2; Z^n, W_1)$.

Consider a random code $C$ for the described transmission. Namely, choose $M_1M_2$ codewords independently, and choose the coordinates of each of the codewords as i.i.d. Bernoulli bits with probability $p(0) = p(1) = 1/2$.

(d) Let $\gamma_n$ be the probability that $Y^n$ does not uniquely determine the pair $(W_1, W_2)$ and let $E\gamma_n$ be the expected value of this probability over the choice of the code $C$. Let $R_i = \frac{1}{n} \log M_i$, $i = 1, 2$. Does the condition $R_1 + R_1 < 1 - p$ imply that $\lim_{n \to \infty} E\gamma_n = 0$?

(e) Let $\delta_n$ be the probability that $(Z^n, W_1)$ does not uniquely determine $W_2$. Does the condition $R_2 < 1 - q$ imply that $\lim_{n \to \infty} E\delta_n = 0$?

(f) Show that for any $R_1 < q - p, R_2 < 1 - q$ and any $\epsilon > 0$ there exists a sufficiently large $n$ and a code $C$ such that $\max(\gamma_n, \delta_n) < \epsilon$.

2. Cover and Thomas, Problems 7.8, 7.12

3. Cover and Thomas, Problem 7.15. It’s too long to solve all of it, but look over it to improve your understanding of jointly typical sequences.