Problem 1 (10pts) Consider a Markov chain with the transition matrix
\[ P = \begin{pmatrix} 1 - 2p & 2p & 0 \\ p & 1 - 2p & p \\ 0 & 2p & 1 - 2p \end{pmatrix}, \quad 0 \leq p \leq 1/2. \]

(a) Classify the states into absorbing, recurrent, transient and identify recurrent classes. Make sure to consider the boundary cases \( p = 0, p = 1/2. \)

(b) Compute the steady-state distribution of the chain. Pay attention to the boundary cases.

(c) Compute the probabilities \( r_{ij}(2) \) of transitioning from \( i \) to \( j \) in 2 steps for all \( i, j = 1, 2, 3. \)

Solution: (a) If \( p = 0 \) then all the states are absorbing, and the chain has 3 “recurrent classes” formed by the states. If \( 0 < p \leq 1/2 \), then all the states are recurrent, and the chain has a single non-periodic recurrent class.

(b) If \( p = 0 \), the chain has 3 steady-state distributions depending on the state \( X_0 \), which are \((1, 0, 0), (0, 1, 0), (0, 0, 1). \) If \( 0 < p \leq 1/2 \), then \( \pi = \pi P \) gives the equations
\[ \begin{align*}
\pi_1 &= \pi_1 (1 - 2p) + \pi_2 p \\
\pi_2 &= \pi_1 \cdot 2p + \pi_2 (1 - 2p) + \pi_3 \cdot 2p \\
\pi_3 &= \pi_2 p + \pi_3 (1 - 2p)
\end{align*} \]
which yield \( \pi_1 = \pi_3 = 1/4, \pi_2 = 1/2. \)

(c) Computing \( P^2 \), we obtain
\[ R(2) = \begin{pmatrix} (1 - 2p)^2 + 2p^2 & 4p(1 - 2p) & 2p^2 \\ 2p(1 - 2p) & 4p^2 + (1 - 2p)^2 & 2p(1 - 2p) \\ 2p^2 & 4p(1 - 2p) & 2p^2 + (1 - 2p)^2 \end{pmatrix}. \]

Problem 2 (10pts) The joint density of RVs \( X \) and \( Y \) is as follows:
\[ f(x, y) = \begin{cases} e^{-(x+y)} & 0 \leq x < \infty, 0 \leq y < \infty \\ 0 & \text{otherwise} \end{cases} \]
Find the PDF of the random variable \( X/Y. \)
Solution:
Compute the CDF of $Z = X/Y$. For $z > 0$,
\[
F_Z(z) = P\left(\frac{X}{Y} \leq z\right) = \int_{X/Y \leq z} e^{-(x+y)} \, dx \, dy = \\
= \int_{y=0}^{\infty} \int_{x=0}^{zy} e^{-(x+y)} \, dx \, dy \\
= \int_{y=0}^{\infty} (1 - e^{-zy})e^{-y} \, dy \\
= \int_{y=0}^{\infty} (1 - e^{-zy})e^{-y} \, dy \\
= \left\{ \left( -e^{-y} + \frac{e^{-y(z+1)}}{z+1} \right) \right\}_{0}^{\infty} = 1 - \frac{1}{z+1}
\]
Now find
\[
f_{X/Y}(z) = \frac{d}{dz} F_Z(z) = \frac{1}{(z+1)^2}, \quad 0 < z < \infty.
\]

**Problem 3** (10 pts) In the tropical forest of Luamo island, thunderstorms occur all year round. They happen at a Poisson rate of 5 per month.

(a) What is the probability that in a given calendar year there are a total of two thunderstorms in January and August combined?

(b) What is the probability that in a given calendar year there are exactly two (not necessarily consecutive) months out of the twelve months that see no thunderstorms at all?

**Solution:** For the Poisson process, let $P(k, t)$ be the (Poisson) probability of $k$ arrivals in time $t$. Then
\[
p_0 \triangleq P(0,1) = e^{-5}, \quad p_1 \triangleq P(1,1) = 5e^{-5}, \quad p_2 \triangleq P(2,1) = \frac{25}{2} e^{-5}.
\]

(a) The probability of the event of interest is
\[
2p_0 p_2 + p_1^2 = 50e^{-10} \quad \text{OR:} \quad P(2,2) = e^{-2 \times 5} \left( \frac{2 \times 5}{2!} \right)^2 = 50e^{-10}.
\]

(b) $P(\text{two dry months}) = \binom{12}{2}e^{-10}(1 - e^{-5})^{10}$.

**Problem 4** (10 pts)
Two students are to meet in the Student Union. If each of them independently arrives at a time uniformly distributed between 12 noon and 1 pm, find the probability that the first to arrive has to wait longer than 10 minutes.

**Solution:** Place the origin at 12 noon, and let $X \sim Unif[0,1], Y \sim Unif[0,1]$. The event of interest is that $\{|X - Y| > 1/6\} = \{Y < X - 1/6\} \cup \{Y > X + 1/6\}$. Since $P(\{Y < X - 1/6\})$ is the area of the isosceles right triangle with legs 5/6, i.e., 25/72, the answer is $2 \times 25/72 = 25/36$.

Or, if you missed the geometric view, compute the integrals:
\[
2P[X + 10 < Y] = 2 \int_{x+10<y} f(x,y) \, dx \, dy
\]
\[
\begin{align*}
&= 2 \int \int_{x+y<10} f_X(x)f_Y(y) \, dx \, dy \\
&= \int_{y=10}^{60} \int_{x=0}^{y-10} \frac{1}{60^2} \, dx \, dy \\
&= \frac{2}{(60)^2} \int_{y=10}^{60} (y-10) \, dy = \frac{25}{36}.
\end{align*}
\]

**Problem 5** (10pts) The distance between the towns T1 and T2 is 11 miles, and there are 10 mile markers on the road from T1 to T2, with readings 1, 2, \ldots, 10. A marker is chosen randomly with a uniform distribution. Let \( D_1 \) be the distance from T1 to the chosen marker, and let \( D_2 \) be the distance from T2 to that same marker.

(a) Are \( D_1 \) and \( D_2 \) positively or negatively correlated? Justify your answer.

(b) Compute the expected value of \( D_1 \times D_2 \).

**Solution:**

(a) If \( D_1 \) increases, then \( D_2 \) decreases, so they are negatively correlated. To justify this formally, write

\[
\rho_{D_1,D_2} = \frac{E[(D_1 - ED_1)(D_2 - ED_2)]}{\sigma_{D_1}\sigma_{D_2}},
\]

and notice that if \( ED_1 = ED_2 = 5.5 \), and the quantities \((D_1 - ED_1)\) and \((D_2 - ED_2)\) always have opposite signs. Therefore, \( \rho_{D_1,D_2} < 0 \).

(b) Let \( X \) be a uniform RV with pmf \( p_X(k) = 0.1, k = 1, \ldots, 10 \) and let \( Z = X(11 - X) \). The pmf of \( Z \) is

\[
p_Z(k) = 0.2, \quad k = 10, 18, 24, 28, 30.
\]

Then \( EZ = 0.2 \times 110 = 22 \).

**Problem 6** (10pts) Suppose that \( n \) cars passed through a certain intersection within an hour, and \( k \) out of them were passenger cars. The probability that any given car is a passenger car is \( p \) and the type of each car is independent of the other cars. Let \( i \) be a number between 1 and \( n \). What is the probability that the \( i \)th car was a passenger car?

**Solution:** Let \( B \) be the event that the \( i \)th car is a passenger car and let \( A \) be the event that \( k \) out of \( n \) cars are passenger. Then

\[
P(B|A) = \frac{P(AB)}{P(A)} = \frac{\binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}.
\]

The answer does not depend on \( p \).