(i) \( S \cap T \cap U \)  

(ii) \( (S \cap T \cap (T \cup U) \cap (S \cup U)) \)  
\[ = (S \cap T)^c \cap (T \cup U)^c \cap (S \cup U)^c \]  

(iii) \( (S \cap T) \cap (T \cup U) \cup (S \cup U) \)  
this subset is highlighted in the Venn diagram in part (ii) 

(iv) \( (S \cap T) \cup (T \cup U) \cup (S \cup U) \)  
\( \setminus \) \( (S \cap T \cap U) \)  
note that the triple intersection is not included 

(v) \( (S \cup T \cup U)^c = S^c \cap T^c \cap U^c \)  

Sample space \( \Omega = 216 \) possible outcomes 

2. \( P(R_1+R_2+R_3 \text{ even}) = \frac{1}{2} \)  
Indeed, for every outcome of the first two rolls, \((R_1, R_2)\), there are 3 out of 6 possible \( R_3 \) that lead to the even sum \( R_1 + R_2 + R_3 \)  
Thus, the total # of even outcomes = 36 \cdot 3 = 108. 
Since every outcome has probability \( \frac{1}{216} \), the answer is \( \frac{108}{216} = \frac{1}{2} \)  
\( P(S + R_2 + R_3 \text{ even}) = \frac{1}{2} \)  
Here we take a subset of \( \Omega \) formed of all 36 pairs \((R_2, R_3)\), of them 18 sum to an even number. 
\( P(R_1+R_2+R_3 = 5) = \frac{6}{216} = \frac{1}{36} \)  

The event in question is as follows: 

\[
\begin{array}{c|ccc}
R_1 & 1 & 1 & 3 & 1 & 2 & 2 \\
R_2 & 1 & 3 & 1 & 2 & 1 & 2 \\
R_3 & 3 & 1 & 1 & 2 & 2 & 1 \\
\end{array}
\]

3. The largest possible value of \( \pi \) is when \( BCA \), then \( \pi = P(B) = 0.6 \). The smallest possible value is found as follows:  
By normalization,  
\[ P(A \setminus B) + P(B \setminus A) + P(\emptyset) \leq 1 \]  
\[ P(A) + P(B) = P(A \setminus B) + P(B \setminus A) + P(\emptyset) \leq 1 + P(\emptyset) \]  
\[ \Rightarrow P(\emptyset) \geq P(A) + P(B) - 1 = 0.35. \]  
Answer: (a) and (d) are always correct
There are 15 possibilities for $R_1 > R_2$, so
\[
P(R_1 > R_2) = \frac{15}{36} = \frac{5}{12}
\]
Similarly $P(R_1 > R_2+2) = \frac{10}{36} = \frac{5}{18}$

5. The sample space $\Omega$ is the unit square.

\[
P(x_1 - x_2 > 0.2) = \frac{\text{area of } \Delta}{\text{area of } \Omega} = 0.32
\]

6. $P(KK) = \frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652} = \frac{3}{663} = \frac{1}{221}$

\[
P(\text{diff. values}) = 1 - P(\text{identical values}) = 1 - P(C_2 = C_1 | C_1) = 1 - \frac{3}{51} = \frac{48}{51} = \frac{16}{17}
\]
1. (a) Let \( A = \{ F_1 = F_2 = \ldots = F_n = T \} \). The event in question is \( A^c \), so
\[
P(A^c) = 1 - P(A) = 1 - 2^{-n}
\]
(b) There are \( \binom{n}{k} \) different ways to choose \( k \) locations of \( H \), so the required probability is \( \binom{n}{k} 2^{-n} \).
(c) Suppose that \( F_1 F_{i+1} F_{i+2} \) is the first place where the outcomes are \( HHH \). If \( i \geq 2 \), then \( F_{i-1} = T \), but then \( (F_{i-1} F_i F_{i+1}) = TTHH \), so \( TTHH \) appears before \( HHH \). Thus, the only way for \( HHH \) to appear earlier is \( i = 1 \), and the probability of that is \( \frac{1}{2} \).
(d) Let \( A_i = \{ F_i = H \} \), \( A = \bigcap_{i=1}^{\infty} A_i \). Then
\[
P(A) = \lim_{n \to \infty} P\left( \bigcap_{i=1}^{n} A_i \right) =
\]
Let \( A_i = \{ F_1 = \ldots = F_i = H, \text{ anything after that} \} \), i.e., the first \( i \) tosses are \( H \).
The event in question is \( A = \bigcap_{i=1}^{\infty} A_i \)
\[
P(A) = \lim_{i \to \infty} P(A_i) = \lim_{i \to \infty} 2^{-i} = 0.
\]
The sample space is \( \Omega = \{ \text{all infinite sequences of H and T} \} \).
The cardinality of \( \Omega \) is the same as of the set of infinite sequences of 0's and 1's, i.e., the same as of the set \( [0,1] \), i.e. same as \( \mathbb{R} \).

2. From the tree diagram
\[
\begin{array}{c}
F_1 \\
0.97 \\
0.03 \\
F_2 \\
0.57 \\
0.43 \\
F_3 \\
0.05 \\
0.95 \\
0.05 \\
\end{array}
\]
\[
p = \text{Prob (faulty switch)} = 0.012 + 0.03 = 0.042
\]
\[
1 - p = 0.958
\]
Among 5 switches, there are 5 possibilities for one to be faulty
\[
\text{Answer: } 5p(1-p)^4 = 5 \cdot 0.015 \cdot 0.958^4 \approx 0.063
\]
3. Compute \( p = P \left( 2^{nd} \text{ card is smaller than 1}^{st} \right) \)

Using the total probability theorem, let \( A_i = \{ 1^{st} \text{ card} = i \}, \ i = 1, 2, .., n \), and write the required probability as

\[
p = \sum_{i=1}^{n} P \left( 2^{nd} < 1^{st} | A_i \right) P(A_i) = \sum_{i=2}^{n} \frac{i-1}{n-1} \cdot \frac{1}{n} = \frac{1}{n(n-1)} \sum_{i=1}^{n-1} i = \frac{1}{n(n-1)} \frac{n(n-1)-1}{2} = \frac{1}{2}
\]

The probability of the event \( \{ 1^{st} < 2^{nd} \} \) is \( 1 - p = \frac{1}{2} \).

4. Let \( U_1 \) be the event that the first urn is selected, and let \( A \) be the event that the 3 marbles drawn are blue. Then

\[
P(U_1A) = \frac{P(U_1) \cdot P(A|U_1)}{P(A)} = \frac{P(A|U_1) \cdot P(U_1)}{P(A|U_1) \cdot P(U_1) + P(A|U_2) \cdot P(U_2)} = \frac{\frac{3}{33} \cdot \frac{3}{2}}{\frac{3}{33} \cdot \frac{3}{2} + 1 \cdot \frac{1}{2}} = \frac{4}{37}
\]

5. Let \( A_i = \{ 6^{th} \text{ sum equals } i \}, \ i = 2, 3, .., 12 \). Let \( B \) be the event that the sum that appears in \( 6^{th} \) roll has not appeared before. We have

\[
P(B) = \sum_{i=2}^{12} P(B|A_i) \cdot P(A_i)
\]

To compute \( P(A_i) \), consider the 36 possible outcomes

\[
P(A_2) = P(A_{12}) = \frac{1}{36}
\]

\[
P(A_3) = P(A_{11}) = \frac{2}{36}
\]

\[
P(A_4) = P(A_{10}) = \frac{3}{36}
\]

\[
P(A_5) = P(A_9) = \frac{4}{36}
\]

\[
P(A_6) = P(A_8) = \frac{5}{36}
\]

We obtain

\[
P(B) = 2 \left[ \left( \frac{35}{36} \right)^5 \left( \frac{1}{36} \right) + \left( \frac{34}{36} \right)^5 \left( \frac{2}{36} \right) + \left( \frac{33}{36} \right)^5 \left( \frac{3}{36} \right) + \left( \frac{32}{36} \right)^5 \left( \frac{4}{36} \right) + \left( \frac{31}{36} \right)^5 \left( \frac{5}{36} \right) \right]
\]

\[
+ \left( \frac{20}{36} \right)^5 \left( \frac{6}{36} \right) \approx 0.56
\]
Problem set 2, Solutions

\[ BC = \left[ \frac{1}{2}, \frac{3}{4} \right] \cup \left[ \frac{1}{2}, 1 \right] \]
\[ AB = \left[ \frac{1}{2}, \frac{3}{4} \right] \]
\[ BU_C = \left[ \frac{1}{4}, 1 \right] \]
\[ AC = \left[ \frac{1}{4}, \frac{1}{2} \right] \cup \{ \frac{3}{4} \} \]

\[ P \cap A \quad ABC = \left\{ \frac{1}{4}, \frac{3}{4} \right\} \]
\[ A \setminus (BU_C) = A = \left[ \frac{1}{4}, \frac{3}{4} \right] \]

(a) \[ P(AB) = \frac{1}{4} ; \quad P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \quad \text{Answer: yes} \]

(b) \[ P(AC) = \frac{1}{4} ; \quad P(A)P(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \quad \text{Answer: yes} \]

(c) \[ P(ABC) = \frac{1}{4} ; \quad P(A)P(BC) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} \quad \text{Answer: no} \]

(d) \[ P(A \setminus (BU_C)) = \frac{1}{2} ; \quad P(A)P(BUC) = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8} \quad \text{Answer: no} \]
(1) Suppose that the tournament has been played, and after that we randomly choose the 2 players in question. There are \( \binom{2^n}{2} \) possible pairs of players of which
\[
2^{n-1} + 2^{n-2} + \ldots + 1 = 2^n - 1
\]
pairs have been actually formed.
In particular, there were \( 2^{n-1} + 2^{n-2} \) pairs in the 1st and 2nd rounds, so the answer to the first question is
\[
\frac{\binom{2^n}{2} - \binom{2^{n-1}}{2}}{2^n(2^n-1)} = \frac{2^{n-2}(1+2)}{2^{n-1}(2^n-1)} = \frac{3}{2(2^n-1)}
\]
Likewise, the other answers are:
\[
P(\text{final or semi-final}) = \frac{3}{2^{n-1}(2^n-1)}
\]
\[
P(\text{never meet}) = \frac{\binom{2^n}{2} - \binom{2^{n-1}}{2}}{\binom{2^n}{2}} = 1 - \frac{1}{2^{n-1}}
\]
(2) Let \( B_0, B_1, B_2 \) be the events that among the 2 removed balls there are 0, 1, 2 blue balls, respectively
\[
P(B_0) = \frac{5}{15} \cdot \frac{4}{14} = \frac{2}{21}; \quad P(B_1) = \frac{5}{15} \cdot \frac{10}{14} + \frac{10}{15} \cdot \frac{5}{14} = \frac{10}{21} > P(B_2) = \frac{10}{15} \cdot \frac{9}{14} = \frac{3}{7}
\]
Let \( A = \{ \text{the ball } R \text{ is blue} \} \). Use the Bayes formula
\[
P(B_2|A) = \frac{P(A|B_2) \cdot P(B_2)}{P(A|B_0) \cdot P(B_0) + P(A|B_1) \cdot P(B_1) + P(A|B_2) \cdot P(B_2)} =
\]
\[
= \frac{8/13 \cdot 9/21}{10/13 \cdot 2/21 + 9/13 \cdot 10/21 + 8/13 \cdot 9/21} = \frac{72}{20+90+72} = \frac{36}{91}
\]
3. \[ p_3 = \frac{(6)(43)}{(49)(6)} \approx 0.018 \quad p_4 = \frac{495}{49} \approx 0.00027 \]

4. \[ p_5 = \frac{(49)}{(6)} \approx 0.000018 \quad p_6 = \frac{1}{49} \approx 0.000010 \]

4. Write the integers 1, 2, ..., 9,99999 in the form

\[
\begin{array}{c}
000001 \\
000002 \\
\vdots \\
999999
\end{array}
\]

if 5 is not allowed, there remain \(9^6\) numbers among this set.

The probability in question is \(1 - \frac{9^6}{10^6} \approx 0.469\)

5. Let \(X_i = (\text{no one departs at step } i)\), \(i = 1, 2, 3\)

The event in question is \(X_1^c X_2^c X_3^c\).

Compute

\[
P(X_1 \cup X_2 \cup X_3) = P(X_1) + P(X_2) + P(X_3) - P(X_1 X_2) - P(X_2 X_3) - P(X_1 X_3) + P(X_1 X_2 X_3)
\]

\[
= \frac{2^6}{3^6} + \frac{2^6}{3^6} + \frac{2^6}{3^6} - \frac{1}{3^6} - \frac{1}{3^6} - \frac{1}{3^6} + 0 = \frac{7}{27}
\]

and find the answer: \(P(X_1^c X_2^c X_3^c) = 1 - P(X_1 \cup X_2 \cup X_3) = \frac{20}{27}\).

6. Suppose we are placing \(r\) marbles in \(m+n\) cells; there are \(\binom{m+n}{r}\) ways of doing this. In the first \(m\) cells there can be \(i = 0, 1, \ldots, r\) marbles, while the last \(n\) cells will contain \(n-i\) marbles. For a given \(i\), there are \(\binom{m}{i}\binom{n}{r-i}\) placements, so

\[
\binom{m+n}{r} = \sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i}
\]
Problem 1. \( Y_1 \) takes values 1, 2, 3, 4, 5, 6 according to the figure.

**PMF**

\[
P_{Y_1}(k) = \begin{cases} 
\frac{1}{36} & k = 1 \\
\frac{3}{36} & k = 2 \\
\frac{5}{36} & k = 3 \\
\frac{7}{36} & k = 4 \\
\frac{9}{36} & k = 5 \\
\frac{11}{36} & k = 6 
\end{cases}
\]

\[
EY_1 = \sum_{k=1}^{6} k P_{Y_1}(k) = \frac{1}{36} (1 + 6 + 15 + 28 + 45 + 66) = \frac{161}{36}
\]

\[
EY_1^2 = \sum_{k=1}^{6} k^2 P_{Y_1}(k) = 791
\]

\[
Var(Y_1) = EY_1^2 - (EY_1)^2 = \frac{791 \cdot 36 - 161^2}{36 \cdot 36} = \frac{2555}{1296}
\]

**PMF**

\[
P_{Y_2}(k) = \begin{cases} 
\frac{1}{36} & k = 1 \\
\frac{11}{36} & k = 2 \\
\frac{9}{36} & k = 3 \\
\frac{7}{36} & k = 4 \\
\frac{5}{36} & k = 5 \\
\frac{3}{36} & k = 6 \\
\frac{1}{36} & k = 6 
\end{cases}
\]

\[
EY_2 = \frac{91}{36}, \quad Var(Y_2) = \frac{2555}{1296}
\]

\[
EY_2^2 = \frac{301}{36}
\]

**PMF**

\[
P_{Y_3}(k) = \begin{cases} 
\frac{1}{36} & k = 1 \\
\frac{2}{36} & k = 2 \\
\frac{3}{36} & k = 3 \\
\frac{4}{36} & k = 4 \\
\frac{5}{36} & k = 5 \\
\frac{6}{36} & k = 6 \\
\frac{7}{36} & k = 7 \\
\frac{8}{36} & k = 8 \\
\frac{9}{36} & k = 9 \\
\frac{10}{36} & k = 10 \\
\frac{11}{36} & k = 11 \\
\frac{12}{36} & k = 12 
\end{cases}
\]

**PMF**

\[
P_{Y_4}(k) = \begin{cases} 
\frac{1}{36} & k = 1 \\
\frac{2}{36} & k = 2 \\
\frac{3}{36} & k = 3 \\
\frac{4}{36} & k = 4 \\
\frac{5}{36} & k = 5 \\
\frac{6}{36} & k = 6 \\
\frac{7}{36} & k = 7 \\
\frac{8}{36} & k = 8 \\
\frac{9}{36} & k = 9 \\
\frac{10}{36} & k = 10 \\
\frac{11}{36} & k = 11 \\
\frac{12}{36} & k = 12 
\end{cases}
\]

Problem 2. The RV \( X \) follows a hypergeometric distribution.

\[
P_X(k) = \frac{\binom{20}{k} \binom{15}{10-k}}{\binom{35}{10}}, \quad k = 0, 1, 2, \ldots, 10 \quad \text{and} \quad P_X(k) = 0 \quad \text{o/w}
\]

Computing the numbers, we obtain:

\[
P_X(k) = \begin{cases} 
0.000016 & k = 0 \\
0.00056 & k = 1 \\
0.00670 & k = 2 \\
0.0400 & k = 3 \\
0.132 & k = 4 \\
0.254 & k = 5 \\
0.288 & k = 6 \\
0.192 & k = 7 \\
0.072 & k = 8 \\
0.014 & k = 9 \\
0.001 & k = 10 
\end{cases}
\]
Problem 3. Let \( Y \) be the dollar amount of coin. Range (\( Y \)) = 15, 11, 10, 6, 2

\[
\begin{array}{c|ccccc}
\text{pmf} & k & 15 & 11 & 10 & 6 & 2 \\
\hline
\text{p}(k) & \frac{2}{15} & \frac{3}{15} & \frac{1}{15} & \frac{6}{15} & \frac{3}{15} \\
\end{array}
\]

(by counting the number of combinations that produces each amount, divided by all \( \binom{6}{2} = 15 \) pairs)

\[
E(Y) = \frac{1}{15} (30+33+10+36+6) = \frac{115}{15} = \frac{23}{3} = 7.666... 
\]

Thus, if the entry fee is at least $7.67, the house on average draws a profit.

Problem 4. Let \( X \) be the number of tosses till the first time we have seen both \( H \) and \( T \), including that time

\[
p_X(k) = \left(\frac{1}{2}\right)^{k-1}, \quad k \geq 2 \quad \text{because we must have either } \underbrace{HHT\ldots TH}_{k-1} \text{ or } \underbrace{THT\ldots TH}_{k-1}
\]

\[
E(X) = \sum_{i=2}^{\infty} i \left(\frac{1}{2}\right)^{i-1}
\]

Let \( x \) be a number such that \( 0 < x < 1 \). We have

\[
\sum_{i=2}^{\infty} i x^{i-1} = \sum_{i=1}^{\infty} i x^{i-1} = \frac{d}{dx} \sum_{i=0}^{\infty} x^i - 1 = \frac{d}{dx} \left(\frac{1}{1-x}\right) - 1 = \frac{1}{(1-x)^2} - 1
\]

Substituting \( x = \frac{1}{2} \), we find \( E(X) = 3 \).

Problem 5. \( \binom{6}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 = \frac{160}{729} \)

Problem 6. Let \( X \) be the RV in question; \( p_X(k) = \frac{2^k}{k!} e^{-\lambda} \).

We have \( p_X(1) = p_X(4) \), i.e., \( e^{-\lambda} = \frac{\lambda^2}{2!} e^{-\lambda} \), i.e. \( \lambda = 3\sqrt{24} \approx 2.88 \)

\[
E(X) = \text{Var}(X) = \lambda
\]
1. First, compute \( P(X \text{ even}) = \frac{1}{16} \left[ (\frac{1}{2})^4 + (\frac{1}{2})^4 + (\frac{1}{2})^4 \right] = \frac{3}{16} = \frac{3}{8} \).

   \( P_x(k | X \text{ even}) = 0 \) for \( k = 1, 3 \)

   \( P_x(k | X \text{ even}) = \frac{1}{P(X \text{ even})} P_x(k) = 2 P_x(k) = \begin{cases} \frac{3}{8} & k = 0, 4 \\ \frac{3}{4} & k = 2 \end{cases} \)

2. Let \( P_x(k), k = 0, 1, \ldots \) be the Poisson (\( \lambda \)) pmf. We have

   \[ \sum_{k \geq 0} p_x(k) = 1 \]

   \[ \sum_{k \text{ even}} p_x(k) = \frac{\sum_{k \text{ even}} e^{-\lambda} (-\lambda)^k}{k!} = e^{-2\lambda} \]

   Adding these equalities, we obtain

   \[ \sum_{k \geq 0} p_x(k) = \frac{1}{2} (1 + e^{-2\lambda}) \]

3. \( \int_1^3 dx / x^2 = \frac{2}{3} \), so by normalization \( c = \frac{3}{2} \)

   \( EX = \frac{3}{\lambda} \int_1^3 x dx = \frac{3}{\lambda} \ln 3 \); \( EX^2 = \frac{3}{\lambda} \cdot 2 = 3 \); \( Var X = 3 - \frac{9}{\lambda} \ln 3 \approx 0.29 \); \( 2\theta X \approx 1.06 \)

   Therefore \( P(|X - EX| < 2\lambda X) \approx P(|X - 1.65| < 1.06) = P(0.59 < X < 2.71) \approx 0.95 \)

4. Again by normalization

   \[ c \int_{-\infty}^{\infty} \frac{d\theta}{\sqrt{1 - \sin^2 \theta}} = c \int_{-\pi/2}^{\pi/2} \frac{d\sin \theta}{\sqrt{1 - \sin^2 \theta}} = c \pi \]

   So \( c = \frac{1}{\pi} \) and

   \[ f_x(x) = \begin{cases} 
   0, & x < 1 \\
   \frac{1}{\sqrt{1 - x^2}}, & 0 < x < 1 \\
   0, & x > 1 \end{cases} \]

   \[ F_x(x) = \begin{cases} 
   0, & x < 1 \\
   \int_{-1}^{x} \frac{1}{\sqrt{1 - \sin^2 \theta}} d\theta / \pi = \int_{-\pi/2}^{\arcsin x} \frac{1}{2} + \frac{1}{2} \arcsin x, & -1 < x < 1 \\
   1, & x > 1 \end{cases} \]

5. \( F_Y(t) = P(X^2 \leq t) = P(-\sqrt{t} \leq X \leq \sqrt{t}) = \frac{\sqrt{t}}{\sqrt{\lambda}}, \quad 0 \leq t < 4; \quad F_Y(t) = 0, \quad t < 0; \quad F_Y(t) = 1, \quad t \geq 4 \)

   \( f_Y(t) = F_Y'(t) = \frac{1}{2\sqrt{\lambda t}}, \quad 0 < t < 4 \) and \( 0 \quad 0/\lambda \)

   \( F_Z(t) = P(X^3 \leq t) = P(X \leq \sqrt[3]{t}) = \frac{3\sqrt[3]{t} + 2}{3}, \quad -8 < t < 8; \quad F_Z(t) = 0, \quad t < -8, \quad F_Z(t) = 1, \quad t > 8 \)

   \( f_Z(t) = F_Z'(t) = \frac{1}{12t^{2/3}}, \quad -8 < t < 8 \) and \( 0 \quad 0/\lambda \)
1. We find

\[ F_X(x) = \begin{cases} 
0 & x < 0 \\
\frac{1}{\sqrt{\pi}} x & 0 \leq x < \frac{\pi}{2} \\
1 & x \geq \frac{\pi}{2}
\end{cases} \quad \text{if } 0 \leq x \leq \frac{\pi}{2} ; \quad 0 \text{ otherwise}
\]

\[ E(\cos 2X) = \int_0^{\frac{\pi}{4}} \frac{\cos 2x}{\sqrt{\pi}} dx = \frac{2}{\pi} \sin 2x \bigg|_0^{\frac{\pi}{4}} = \frac{2}{\pi}
\]

\[ E(\cos^4X) = E(\frac{1}{8} (1+\cos 2X)) = \frac{1}{8} + \frac{1}{8}
\]

2. \[ F_Y(y) = P(X \leq \tan x) = \int_{-\infty}^{\tan x} \frac{\sin u}{\pi(1+u^2)} \frac{\tan x}{\pi} + 1\]

\[ f_Y(y) = \frac{1}{\pi} \quad \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \quad \text{and} \quad 0 \text{ otherwise}
\]

3. Write \( f_X(x) \) as a PDF of a normal RV:

\[ f_X(x) = \frac{1}{\sqrt{\gamma_2} \pi} e^{-\frac{(x-\mu)^2}{2\gamma_2}} \]

This shows that \( EX = 1, \ Var X = \frac{1}{4} \)

4. \[ P(Y \leq X) = \frac{1}{2}; \quad P(X^2 + Y^2 \leq 1) = \frac{\pi}{4} \]

by calculating the areas of the regions \( Y \leq X \) and \( X^2 + Y^2 \leq 1 \).

5. To find the marginal PDFs, compute

\[ f_X(x) = \int_0^\infty 8xy dy = 4x^3 \quad 0 \leq x \leq 1 \quad f_Y(y) = \int_0^\infty 8xy dy = 4y (1-y^2) \quad 0 \leq y \leq 1 \]

\[ f_X(x)f_Y(y) = 16x^3y(1-y^2); \quad f_{XY}(x,y) = 8xy, \quad \text{so } X \text{ and } Y \text{ are not independent}
\]

6. Let \( Z = \max(X,Y) \)

\[ F_Z(z) = P(\max(X,Y) \leq z) = P(X \leq z \land Y \leq z) = P(X \leq z)P(Y \leq z) = (1-e^{-\lambda z})^2, \quad z \geq 0
\]

by independence

\[ f_Z(z) = F'_Z(z) = 2\lambda e^{-\lambda z} (1-e^{-\lambda z}) = 2\lambda e^{-2\lambda z} - 2\lambda e^{-\lambda z}, \quad z \geq 0
\]

\[ EZ = \int_0^\infty z E[Z] = 2\lambda \left[ -\frac{1}{\lambda} e^{-\lambda z} \right]_0^\infty + \frac{1}{\lambda} \int_0^\infty e^{-2\lambda z} dz + \frac{1}{2\lambda} \int_0^\infty e^{-\lambda z} dz + \frac{1}{2\lambda} \int_0^\infty e^{-2\lambda z} dz \]

\[ = \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda}
\]

Let us find \( f_U(u) \), where \( U = \frac{X}{X+Y} \). Put \( V = X \)

\[ Range(V) = (0,\infty), \quad Range(U) = (0,1)
\]
We have \( x = v^* \), \( y = \frac{v}{u^*} - u^* \), \( J = \begin{vmatrix} 0 & 1 \\ -\frac{v}{u^*} & 1 \end{vmatrix} = \frac{v}{u^*} \)

Since \( f_{XY}(x, y) = \lambda^2 e^{-\lambda(x+y)} \), \( 0 < x, y < \infty \), we obtain

\[
\begin{align*}
f_{UV}(u, v) &= \lambda^2 \frac{v}{u^*} e^{-\lambda(v + \frac{v}{u^*} - u^*)} = \lambda^2 \frac{v}{u^*} e^{-\lambda \frac{v}{u^*}} \quad 0 < v < \infty, \quad 0 < u < 1 \\
f_U(u) &= \int_0^\infty f_{UV}(u, v) \, dv = \frac{\lambda^2}{u^*} \int_0^\infty v e^{-\frac{v}{u^*}} \, dv = \frac{\lambda^2}{u^*} \int_0^\infty \frac{v}{u^*} e^{-\frac{v}{u^*}} \, dv = 1, \quad 0 < u < 1 \\
U &\sim \text{unif}[0, 1]
\end{align*}
\]

**Problem 7**

\[
P(X < 1 | Y = 3) = \int_0^1 f_{X|Y}(x|3) \, dx = \int_0^1 \frac{x+3}{4} e^{-x} \, dx = \frac{1}{4} \left[ \frac{x e^{-x}}{3} + 3 \int e^{-x} \, dx \right]_{0}^{1} = -\frac{1}{4} \left[ e^{-x} \right]_{0}^{1} - \frac{3e^{-x}}{4} \bigg|_0^1 = -\frac{1}{4} e - \frac{3}{4} e + \frac{3}{2} = \frac{13}{4} \frac{1 + \frac{1}{e} - \frac{3}{2} e}{4e} = 1 - \frac{5}{4e}
\]

**Problem 8**

\[
f_{XY}(x, y) = 2, \quad x > 0, \quad x+y < 1 \quad \text{and} \quad 0 < y < 1
\]

\[f_{XY}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{2}{1-y}, \quad 0 < x < 1-y\]

\[E(X|Y=y) = \frac{1}{1-y} \int_0^{1-y} x \, dx = \frac{1-y}{2y}\]

**Problem 9**

We have

\[
x = r \cos \theta \quad y = r \sin \theta \\
J = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r
\]

\[
f_{R\Theta}(r, \theta) = r f_{XY}(r \cos \theta, r \sin \theta) = \frac{r}{2\pi} e^{-r^2/2}, \quad 0 < \theta < 2\pi, \quad r > 0
\]

\[f_R(r) = \int_0^{2\pi} f_{R\Theta}(r, \theta) \, d\theta = re^{-r^2/2}, \quad r > 0
\]

\[f_\Theta(\theta) = \int_0^\infty \frac{re^{-r^2/2}}{2\pi} \, dr = \frac{1}{2\pi}, \quad 0 < \theta < 2\pi
\]

\[f_{R\Theta}(r, \theta) = f_R(r) f_\Theta(\theta), \quad \text{so} \quad R \perp \perp \Theta.
\]
1. Let \( X_i = \begin{cases} 1 & \text{if roll is 2} \\ 0 & \text{else} \end{cases} \quad Y_i = \begin{cases} 1 & \text{if roll is 3} \\ 0 & \text{else} \end{cases} \quad i = 1, 2, \ldots, n \\
We have \( X = \sum_{i=1}^{n} X_i \quad Y = \sum_{i=1}^{n} Y_i \)
\[
\text{cov}(X, Y) = \text{cov} \left( \sum_{i=1}^{n} X_i, \sum_{j=1}^{n} Y_j \right) = \sum_{i,j=1}^{n} (E(X_i Y_j) - E(X_i)E(Y_j))
\]
If \( i \neq j \), then \( X_i \) and \( Y_j \) are independent, so \( \text{cov}(X_i, Y_j) = 0 \). Continue as follows:
\[
= \sum_{i=1}^{n} \left( E(X_i Y_i) - E(X_i)E(Y_i) \right) = \sum_{i=1}^{n} \left( 0 - \frac{1}{6} \cdot \frac{1}{6} \right) = -\frac{n}{36}
\]
Note that if \( X \) is greater than \( Y \) is likely to be smaller, so \( X \) and \( Y \)
are negatively correlated (\( \text{cov}(X, Y) < 0 \)).

2. We have \( f_X(x) = 1 \quad 0 < x < 1 \) and \( 0 \text{ of } \Omega \). \( E_X = 1/2 \), \( E_X^2 = 1/3 \), \( E_X^3 = 1/6 \), \( E_X^4 = 1/5 \), \( E_X^5 = 1/4 \)
\[
\sigma_X^2 = E(X^2) - (E_X)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}; \quad \sigma_Y^2 = E(Y^2) - (E_Y)^2 = E(X^4) - (E_X^4)^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}
\]
\[
\text{cov}(X, Y) = E(X^2) - E(X)E(X) = E(X^4) - (E_X^4)^2 = \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{12}
\]
\[
P(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\frac{1}{12}}{\frac{1}{4} \cdot \frac{\sqrt{5}}{12}} = \frac{\sqrt{5}}{12} = \frac{\sqrt{5}}{12}
\]

3. (a) Find \( E(X_i | X_{i-1} = m) = \frac{m}{w+b} \cdot m + \frac{w+b-m}{w+b} \cdot (m+1) = \frac{m}{w+b} \cdot m + \left(1 - \frac{m}{w+b} \right) (m+1) \)
\[
= 1 + \left(1 - \frac{m}{w+b} \right) m, \quad i \geq 1
\]
Thus \( E(X_i | X_{i-1}) = 1 + \left(1 - \frac{m}{w+b} \right) X_{i-1} \)
Taking the expectation on both sides, we obtain
\[
E(E(X_i | X_{i-1})) = EX_i = 1 + \left(1 - \frac{1}{w+b} \right) EX_{i-1}, \quad i \geq 1
\]
see next page
4. From the formulas provided, we observe that the conditional PDF of $Y$ conditioned on $X=3.6$ is Gaussian with mean

$$\mu_Y + \varphi \frac{\sigma_Y}{\sigma_X} (x-\mu_X) = 2.5 + 0.4 \frac{0.4}{0.5} (3.6 - 3) = 2.692$$

and variance

$$\sigma^2 = \sigma_Y \sqrt{1-p^2} = 0.4 \sqrt{0.84} \approx 0.37$$

Then, denoting $Z \sim N(0,1)$ a standard normal RV, we obtain

$$P(Y \geq 3 \mid X=3.6) = P \left( \frac{Y - 2.692}{0.37} \geq \frac{3 - 2.692}{0.37} \mid X=3.6 \right) = P(Z \geq 0.832)$$

$$= 1 - \Phi(0.832) \approx 1 - 0.7977 = 0.203$$

3. Let $X_i = \# \text{ blue balls after } i \text{ trials}, i=1,2,...,n$. Suppose that $X_i=m$, then

$$E(X_i \mid X_{i-1}=m) = \frac{m}{b} + \frac{b-m}{b} (m+1) = 1 + \left(1 - \frac{1}{b}\right) m, \ i \geq 1 \ (\text{if } i=1, \text{then the condition is } X_0 = b)$$

In particular

$$E X_1 = 1 + \left(1 - \frac{1}{b}\right) b = B - \pi \left(1 - \frac{1}{b}\right)$$

Generally we find (using)

$$E(X_i \mid X_{i-1}) = 1 + \left(1 - \frac{1}{b}\right) X_{i-1}, \ i \geq 1$$

Compute the expectation of this relation, recalling that $E(E(X_i \mid X_{i-1})) = EX_i$:

(A) $EX_i = 1 + \left(1 - \frac{1}{b}\right) EX_{i-1}, \ i \geq 1$

Now take $i=n$ and use (A) repeatedly $n-1$ times:

$$EX_n = 1 + \left(1 - \frac{1}{b}\right) EX_{n-1} = 1 + \left(1 - \frac{1}{b}\right) \left(1 + \left(1 - \frac{1}{b}\right) EX_{n-2}\right)$$

$$= 1 + \left(1 - \frac{1}{b}\right) + \left(1 - \frac{1}{b}\right)^2 \left(1 + \left(1 - \frac{1}{b}\right) EX_{n-3}\right)$$

$$= 1 + \left(1 - \frac{1}{b}\right) + \left(1 - \frac{1}{b}\right)^2 + \left(1 - \frac{1}{b}\right)^3 \left(1 - \frac{1}{b}\right)^n EX_1$$

$$= B - B \left(1 - \frac{1}{b}\right)^n + \left(1 - \frac{1}{b}\right)(B - \pi \left(1 - \frac{1}{b}\right))$$

$$= B - \pi \left(1 - \frac{1}{b}\right)^n$$
5) We have 
\[ u = x + y \quad \text{so} \quad x = \frac{1}{2} (u + v) \]
\[ v = x - y \quad \text{so} \quad y = \frac{1}{2} (u - v) \]
\[ J = \frac{1}{2} \]

\[ f_{uv}(u,v) = f_{xy}(x+y, x-y) \cdot J \]

and we conclude that the pair \( U, V \) follows the bivariate normal distribution. Further
\[ \text{cov}(U; V) = \text{cov}(X+Y, X-Y) = \text{var}(X) - \text{var}(Y) = 0 \]

so \( U \) and \( V \) are uncorrelated. As shown in class, two uncorrelated Gaussian RVs are independent, so the answer is Yes. (see also p.256 of the textbook)

6) a) Since
\[ M_x(s) = \sum p_x(k) e^{sk} \]
we see that \( p_x(k) = 0 \) if \( k = 1, 3, 4, 7 \). By the uniqueness of transform property we find the PMF
\[ p_x(k) = \frac{1}{10}, \frac{3}{10}, \frac{5}{10}, \frac{1}{10} \]
\[ k = 1, 3, 4, 7 \]
and from this compute \( P(X \leq 5) = \frac{9}{10} \)

(6) Given \( M_x(s) = e^{2s^2} \)

The transform of a Gaussian \( N(\mu, \sigma^2) \) is \( e^{\mu s + \frac{1}{2} s^2 \sigma^2} \), so using the uniqueness of transforms property, we conclude that each RV is Gaussian with mean \( \mu = 0 \) and \( \sigma^2 = 2 \), i.e., \( \sigma = \sqrt{2} \).

Now find
\[ P(0 < X < 1) = P(0 < \frac{X - \mu}{\sigma} < \frac{1 - \mu}{\sigma}) = P(0 < Z < \frac{1}{\sqrt{2}}) \]
where \( Z \sim N(0, 1) \)
\[ = \Phi\left(\frac{1}{\sqrt{2}}\right) - \Phi(0) = 0.69 - 0.5 = 0.19 \]
Problem 1. From the table on p. 239,

\[ X \sim \text{Poisson}(2) \quad \Pr(X=i) = e^{-2} \frac{2^i}{i!}, \quad i=0,1,2, \ldots \quad \text{EX}=2 \]

\[ Y \sim \text{Binom}(10, \frac{3}{4}) \quad \Pr(Y=0) = \frac{1}{4^{10}}, \quad \Pr(Y=1) = \frac{3}{4} \times \frac{1}{4^9}, \quad \Pr(Y=2) = \frac{1}{16} \times \frac{1}{4^8} \]

The below calculation is for \(X+Y=2\); the calculation for \(X+Y=3\) is analogous

\[
P(X+Y=2) = P(X=0, Y=2) + P(X=1, Y=1) + P(X=2, Y=0)\]

\[
= \frac{1}{10} \cdot e^{-2} \cdot \frac{1}{4^{10}} + 10 \cdot \frac{3}{4} \cdot e^{-2} \cdot \frac{1}{4^9} + 45 \cdot \frac{1}{4} \cdot e^{-2} \cdot \frac{1}{4^8} = e^{-2} \left( 2 \cdot 60 + 405 \right) = 467 \cdot \frac{e^{-2}}{4^{10}}
\]

\[
P(XY=0) = P(X=0 \quad \text{or} \quad Y=0) = P(X=0, Y=0) + P(X=0, Y>0) + P(X>0, Y=0)
\]

\[
= P(X=0) + P(Y=0) - P(X=0, Y=0) = e^{-2} + 4^{-10} - e^{-2} \cdot 4^{-10} = e^{-2} (1 - 4^{-10}) + 4^{-10}
\]

\[
\text{E}XY = \text{EX} \cdot \text{EY} = 2 \cdot \left( 10 \cdot \frac{3}{4} \right) = 15
\]

Problem 2. Let \(Y = \sum_{i=1}^{10} X_i\); we have \(\text{EX}_i = 1; \text{Var}(X_i) = 1; \text{EY} = 20\)

(a) \(P(Y > 25) < \frac{\text{EY}}{25} = \frac{4}{5}\) using the Markov inequality

(b) using the CLT,

\[
P\left(\frac{Y-20}{\sqrt{20}} > \frac{25-20}{\sqrt{20}}\right) \approx P\left(\frac{Y-20}{\sqrt{20}} > 1.12\right) \approx 1 - \Phi(1.12) \approx 0.1314
\]

Problem 3. Let \(X = \sum_{i=1}^{10000} X_i\); \(\mu = 240, \eta \mu = 10^4, 240 = 24 \cdot 10^6, \sigma \sqrt{n} = 8.10^4\)

\[
P\left(\frac{X-24 \cdot 10^6}{8 \cdot 10^4} > \frac{2.7 \cdot 10^6 - 24 \cdot 10^6}{8 \cdot 10^4}\right) \approx P(Z > 3.75), \text{ where } Z \sim N(0,1)
\]

So the answer is \(1 - \Phi(3.75) \approx 1 - 0.999912 = 0.000088\).
Problem 4. Choose $\varepsilon > 0$. Frankly it looks as though $Y_i, i = 1, 2, \ldots$ converge to 0, so let us check this possibility.

$$P(|Y_i - 0| > \varepsilon) = 1 - P(|Y_i| \leq \varepsilon) = 1 - P(X \leq i) = 1 - \int_{0}^{i} f_X(x) dx$$

As $i \to \infty$, the last integral approaches 0, so

$$\lim_{i \to \infty} P(|Y_i| > \varepsilon) = 0,$$

which shows that the sequence $Y_i$ converges to 0 in probability.
Problem 5

(a) \( P(N(\frac{1}{3}) = 2) = \left(\frac{1}{3} \lambda\right)^2 \frac{e^{-\lambda/3}}{2!} = \frac{1}{18} \lambda^2 e^{-\lambda/3} \)

\[ P(N(\frac{1}{3}) = 2 \mid N(1) = 2) = \frac{P(N(\frac{1}{3}) = 2) P(N(\frac{1}{3}) = 0)}{P(N(1) = 2)} = \frac{(\frac{1}{3})^2 \lambda^3 e^{-\lambda/3} \cdot e^{-\lambda/3}}{\lambda^2 e^{-\lambda/2}} = \frac{1}{9} \]

(b) \( \Pr(N(\frac{1}{3}) = 2) = \Pr(N(1) = 2) - \Pr(N(\frac{1}{3}) = 0 \mid N(1) = 2) = 1 - \frac{P(N(\frac{1}{3}) = 0) P(N(\frac{1}{3}) = 2)}{P(N(1) = 2)} \)

\[ = 1 - \frac{e^{-\lambda/3} \cdot (\lambda/3)^2 e^{-2\lambda/3}}{2 \cdot \lambda^2 e^{-\lambda/2}} = 1 - \frac{4}{9} = \frac{5}{9} \]

(c) We need to find \( P(N(s) = i \mid N(t) = n), \ 0 < i < n \)

\[ P(N(s) = i \mid N(t) = n) = \frac{P(N(s) = i, N(t) = n)}{P(N(t) = n)} = \frac{P(N(s) = i, N(t-s) = n-i)}{P(N(t) = n)} \]

\[ = \frac{P(N(s) = i) P(N(t-s) = n-i)}{P(N(t) = n)} = \frac{n!}{i!(n-i)!} \cdot \frac{e^{-\lambda s} (\lambda s)^i e^{-\lambda(t-s)} (\lambda(t-s))^{n-i}}{e^{-\lambda t} (\lambda t)^n} \]

\[ = \frac{n!}{i!(n-i)!} \left(\frac{s}{t}\right)^i \left(1 - \frac{s}{t}\right)^{n-i} \sim \text{Binom}(n, \frac{s}{t}) \]

Problem 6. The error process is Poisson with \( \lambda = \frac{4}{60} \). Let \( N(t) \) be the number of errors in \( t \) transmissions, so \( N(t) \sim \text{Poisson}(\lambda t) \).

In 10 sec. there are 70 transmissions, and we would like to find \( P(N(70) > 1) \):

\[ P(N(70) > 1) = 1 - P(N(70) = 0) - P(N(70) = 1) = 1 - e^{-70/60} - \frac{70}{60} e^{-70/60} = 1 - \frac{13}{6} e^{-7/6} \]

\[ \approx 0.325 \]
Prob. 1. \( R(n) = P^n \), where
\[
\begin{bmatrix}
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Computing \( r_1(4) \), we find \( r_1(4) = \frac{3}{8} \)

\[
\frac{r_{12}(5)}{r_{12}(6)} + \frac{r_{16}(5)}{r_{16}(6)} = \frac{11}{16} + \frac{3}{16} = \frac{11}{16}
\]

Prob. 2. By inspection, 2 is transient, and \( \{1,4\} \) and \( \{3,5\} \) form disjoint recurrent classes.

Prob. 3. The 8-state chain has 8 states, 1, 2, 3, corresponding to the number of the player that is currently drawing the card. The transitions are given by the following matrix:

\[
P = \begin{bmatrix}
1/8 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Solving the system \( \pi P = \pi \), where \( \pi = (\pi_1, \pi_2, \pi_3) \), we obtain

\[
\pi_1 = \frac{39}{64}, \; \pi_2 = \frac{12}{64}, \; \pi_3 = \frac{13}{64}
\]

Prob. 4. Let \( x_i = \text{Prob. (The chain is absorbed into 4 | start from i)}, i=1,2,...,7. \)

Using the total probability theorem, we obtain

\[
x_1 = 0.3x_1 + 0.7x_2 \\
x_2 = 0.3x_1 + 0.2x_2 + 0.5x_3 \\
x_3 = 0.6x_4 + 0.4x_5 \\
x_4 = 1 \\
x_5 = (1-x)/3 \\
x_6 = 0.1x_1 + 0.3x_2 + 0.1x_3 + 0.2x_5 + 0.6x_6 + 0.1x_7 \\
x_7 = 0
\]

\[
\begin{align*}
x_1 &= x_2 = x_3 = x_4 = x_5 = 1 \\
x_6 &= 0.875 \\
x_7 &= 0
\end{align*}
\]

Answer: 0.875

Prob. 5. Use the total expectation theorem. 
Average count till 1 till 2 till 3
\[
t_2 = 1 + 0.15t_2 + 0.53t_3 \\
t_3 = 1 + 0.13t_2 + 0.27t_3 \\
t_2 = 2.28, \; t_3 = 1.77 \\
\text{Ans: 1.67}
\]

\[
t_1 = 1 + 0.2t_1 + 0.5t_3 \\
t_3 = 1 + 0.6t_1 + 0.27t_3 \\
t_1 = 4.33, \; t_3 = 4.92
\]

\[
\text{Ans: 3.38}
\]

\[
t_1 = 1 + 0.2t_1 + 0.3t_2 \\
t_2 = 1 + 0.32t_1 + 0.15t_2 \\
t_1 = 1.97, \; t_2 = 1.92
\]

\[
\text{Ans: 1.61}
\]

\[
\frac{1}{(t_1+t_2+t_3)}
\]
Let $i = \# \text{ of chargers at the current location (home or office)}, i = 0, 1, \ldots, r$. Then $p_{i,i+1} = 1-p$; $p_{i,i} = p$, for $i = 1, 2, \ldots, r$, and $p_{0r} = 1$.

$$P = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & p & 1-p \\ 0 & 0 & 0 & \cdots & p & 1-p & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & p & 1-p & 0 & 0 & 0 \\ p & 1-p & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

Balance equations:

\[
\begin{align*}
\pi_0 &= p\pi_1 \\
\pi_1 &= p\pi_2 + (1-p)\pi_r \\
\pi_2 &= p\pi_3 + (1-p)\pi_{r-1} \\
& \vdots \\
\pi_{r-1} &= p\pi_r + (1-p)\pi_1 \\
\pi_r &= \pi_0 + (1-p)\pi_1
\end{align*}
\]

Using the first and the last equations, we obtain $\pi_r = \pi_1$.

Then, next to the last one gives $\pi_{r-1} = \pi_r$, and generally

\[
\pi_1 = \pi_r = \cdots = \pi_{r-1} = \pi_r = \frac{1}{p} \frac{\pi_0}{p}
\]

(a) So $1 = \pi_0 + \pi_1 + \cdots + \pi_r = \pi_r \left( \frac{p}{1} + 1 + \cdots + 1 \right) \Rightarrow \pi_1 = \cdots = \pi_r = \frac{1}{p+1}; \quad \pi_0 = \frac{p}{p+1}$

(b) For $r = 3$, the proportion of time in state 0 is $\pi_0 = \frac{p}{p+2}$

(c) $\frac{p}{p+2}$ is maximal when $p = 1$. 