1. Let \( b(n, k, p) = \binom{n}{k} p^k (1-p)^{n-k} \) denote the binomial probability

\[
P(\text{player 1 wins}) = 1 - b(6, 0, \frac{5}{6}) = 1 - \left(\frac{5}{6}\right)^6 \approx 0.665
\]

\[
P(\text{player 2 wins}) = 1 - b(12, 0, \frac{5}{6}) - b(12, 1, \frac{5}{6}) = 1 - \left(\frac{5}{6}\right)^{12} - 12 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{11} \approx 0.619
\]

Answer: player 1.

2. Answer \( \frac{\binom{10}{5}}{\binom{100}{10}} \approx 0.3305 \)

3. (a) Interesting question. If the marbles are indistinguishable, then there is just one way to partition them into pairs because every placement results in the same outcome.

If the marbles are numbered 1, 2, ..., 8, then there are

\[
\frac{1}{4!} \binom{8}{2, 2, 2, 2} = \frac{1}{4!} \cdot \frac{8!}{(2!)^4} = \frac{2520}{24} = 105 \text{ ways}
\]

Either answer is acceptable if supplied with correct reasoning.

(b) Every lineup has the same probability, so the needed quantity

\[
\frac{\# \text{ alternating lineups}}{\# \text{ all possible lineups}} = \frac{3 \cdot 3 \cdot 2 \cdot 2 \cdot 2}{6!} = \frac{72}{720} = 0.1
\]

4. Under the uniform distribution of quintuples, the needed quantity is

\[
\frac{\binom{15}{5}}{\binom{20}{5}} \approx 0.19
\]

5. Let us write the numbers as 000, 001, ..., 999. Let \( A_i \) be the subset of numbers that contain 1 in position \( i = 1, 2, 3 \).

The needed quantity is

\[
|A_1 \cup A_2 \cup A_3| - |A_1 A_2| - |A_1 A_3| - |A_2 A_3| + |A_1 A_2 A_3|
\]

\[
= 3 \cdot 100 - 3 \cdot 10 + 1 = 271 = \# \text{ numbers with at least one 1}
\]

\[
\text{Prob (random number contains \( \geq \) one 1)} = \frac{271}{1000} = 0.271
\]
6) Let \( b \) = \# blue balls in the first urn

\[ r = \# \text{ red balls in the first urn} \]

By the total prob. theorem

\[ P(\text{the chosen ball is blue}) = \frac{1}{2} \frac{b}{b+r} + \frac{1}{2} \frac{5-b}{(5-b)+(6-r)} \]

This number is maximal when \( b=1, r=0 \)

and equals 0.7222...

Answer: place a single blue ball and no red balls in one of the urns.
Let $p = P(a \text{ random point } \leq \frac{1}{4}) = \frac{1}{4}$

**Answer:** $1 - 5p(1-p)^4 - (1-p)^5 = 1 - 5 \cdot \frac{3}{4}^4 - \frac{3}{4}^5 \approx 0.367$

2. $n = 3000; \quad p = 0.0005$
   
   The parameter of the Poisson distribution, $\lambda = np = 1.5$

   \[ \text{Prob (0 or 1 or 2 carry the mutation)} \approx e^{-1.5} \left(1 + \lambda + \frac{\lambda^2}{2}\right) = 3.365 \cdot e^{-1.5} \approx 0.751 \]

3. $P(1 \text{ or more } \geq 6) = 1 - P(\text{none are equal to 6}) = 1 - 0.9^n = 0.9^n \leq 0.1 \Rightarrow n \geq 22$

4. The probability that both players get the same number of $H$ equals

   \[ \sum \binom{n}{i}^2 \left(\frac{1}{4}\right)^i \left(\frac{1}{4}\right)^{n-i} = 2^{-2n} \sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n} \cdot 2^{-2n} \]

   To compute $\sum \binom{n}{i}^2$ observe that

   \[ \binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i} \binom{n}{n-i} = \sum_{i=0}^{n} \binom{n}{i}^2 \]

5. Let $F$ be the failure event. We know that

   $P(F|A) = 0.2 \quad P(F|B) = 0.3 \quad P(F|C) = 0.36$

   Use the Bayes formula to compute $P(C|F)$:

   \[ P(C|F) = \frac{P(F|C) P(C)}{P(F|A) P(A) + P(F|B) P(B) + P(F|C) P(C)} \]

   \[ = \frac{0.36 \cdot 0.2}{0.2 \cdot 0.8 + 0.3 \cdot 0.5 + 0.36 \cdot 0.2} = \frac{0.072}{0.66 + 0.15 + 0.072} \approx 0.188 \]

   \[ \approx 0.382 \]
\[ E \sum_{i=1}^{10} 0.1 \pi i^2 = 0.1 \pi \sum_{i=1}^{10} i^2 = 0.1 \pi \frac{10 \cdot 11 \cdot 21}{6} \approx 38.5 \pi \]

\[ \approx 120.951 \]

7. Suppose that the first time we have seen both \( H \) and \( T \) is \( n \). This can happen only if the sequence of tosses up to (and including) toss \( n \) is

\[ A_n \left\{ TTT \ldots TH \right\} \]

Any other sequence of outcomes would have resulted in getting both \( H \) and \( T \) earlier than \( n \).

The probability \( P(A_n) = \frac{2}{2^n} = 2^{-n+1} \)

\[ E[N] = \sum_{n\geq2} n P(A_n) = 2 \left( \frac{1}{4} + \frac{1}{8} + \ldots \right) = \frac{2}{3} - 1 + \sum_{n=1}^{\infty} \frac{n}{2^n} \]

\[ = -1 + \frac{1}{(1-\frac{1}{2})^2} = 3 \]
1a. Let \( X \) be the RV equal to the number of cards before the 1st ace.

\[
P(X = i) = \frac{\binom{48}{i-1}}{\binom{52}{i-1}} \frac{4}{52 - (i-1)}
\]

\( \text{no aces among the first } i-1 \) opened on the first \( i+1 \) opened

\[
E[X] = \sum_{i=1}^{49} i \cdot P(X=i) \approx (\text{computer}) [10.6]
\]

1b. Let \( X \) be defined as in part 1a.

\( X \sim \text{geom}(p) \), where \( p = \frac{4}{52} = \frac{1}{13} \)

\[
P(X \geq q) = \frac{\binom{12}{q-1}}{\binom{13}{q-1}} = \left(\frac{12}{13}\right)^q \approx 0.49
\]

\( \text{at least } q \text{ tries before the first success} \)

2. \[
E[X(X-2)] = E[X^2 - 2X] = EX^2 - 2EX = EX^2 - 2 = 3
\]

\[
\Rightarrow EX^2 = 5
\]

\[
\text{Var}(X) = EX^2 - \left(\frac{EX}{2}\right)^2 = 4 ; \ Var(-3X+5) = 9 \text{Var}(X) = 36
\]

3. If the fellow is courteous enough to return the coats to the hanger, then his probability of success is \( \frac{1}{n} \) every time, and the success is described by a geometric RV \( (p = \frac{1}{n}) \). The expectation \( EX = n \), \( \text{Var}(X) = \frac{1-p}{p^2} = \frac{n}{n-1} \)

If he's a true boor, and tosses the coats on the floor, then his probability of success in trial \( i \) is

\[
P_1 = \frac{1}{n} ; P_2 = \frac{1}{n-1} \frac{n-1}{n} = \frac{1}{n} ; \ldots ; P_i = \frac{1}{n-(i-1)} \times \frac{n-(i-1)}{n} = \frac{1}{n}
\]

\( \text{finally, a success} \)

\[
\text{no luck in the first } (i-1)
\]

Thus, his success time is uniformly distributed between \( \{1, 2, \ldots, n\} \), so

\[
E = \frac{n+1}{2} ; \ Var = \frac{n^2-1}{12}
\]
Use the provided "success" probability, i.e., $p = 0.03$ of taking more than 5 years.

4. \text{Prob (2, out of 24, took more than 5 years)}

\[
1 - (\binom{24}{2}) (0.97)^{24} - (\binom{24}{1}) 0.03 \times 0.97^{23} \approx 0.162
\]

Thus the claim that 3% take more than 5 years looks probable (the computed number is <50%). It is not fair to reject the claim.

5. \[
P_Y(0) = \sum_{x=0}^{\infty} P_{xY}(x, 0) = \frac{1}{2} (1-p) + \frac{1}{2} p = \frac{1}{2}
\]

\[
P_Y(1) = 1 - P_Y(0) = \frac{1}{2}
\]

\[
P_{Y|x}(0|1) = \frac{P_{X,Y}(1, 0)}{P_x(1)} = \frac{\frac{1}{2} (1-p)}{\frac{1}{2}} = \frac{1}{2} (1-p)
\]

\[
P_{Y|x}(0|0) = \frac{P_{X,Y}(0, 0)}{P_x(0)} = \frac{\frac{1}{2} (1-p)}{\frac{1}{2}} = \frac{1}{2} (1-p) = 1 - P_Y(0)
\]

6. (We are assuming that X and Y are independent because we do not have any information to conclude otherwise)

(a) \[
P(X + Y = 10) = \sum_{i=1}^{9} P(X=i, Y=10-i) = \sum_{i=1}^{9} P(X=i) P(Y=10-i)
\]

\[
= \sum_{i=1}^{9} p (1-p)^{i-1} \times p (1-p)^{10-i-1} = \frac{9}{2} \sum_{i=1}^{9} p^2 (1-p)^8 = 9 p^2 (1-p)^8
\]

(b) \[
P_{X|Y}(i|10) = \frac{P(\{X=i\} \cap \{X+Y=10\})}{P(X+Y=10)} = \frac{P(\{X=i\}) P(\{Y=10-i\})}{9 p^2 (1-p)^8}
\]

\[
= \frac{p^2 (1-p)^8}{9 p^2 (1-p)^8} = \frac{1}{9} \quad \text{for all } i = 1, 2, \ldots, 9
\]

\[
P_{X|Y}(i|10) = 0 \quad \text{for all } i \leq 0 \text{ or } i > 10
\]
1. \[ X \begin{array}{cccc} 0 & 1 & 2 & 3 & 4 \\
\end{array} \]

\[ P_X(i) = \frac{(30)(\text{P}^i)(1-\text{P})^{4-i}}{(52)(\text{P})^i}, \quad i = 0, 1, 2, 3, 4 \]

\[ \text{EX} = \frac{30(\text{P})^2(22)(\text{P})^{4-2}}{(52)(\text{P})^2} \left[ \frac{1}{(\text{P})^4} \right] \approx 2.3079 \]

2. First, find \( a \).

\[ \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} = e^{2t} \]

So, \( a = e^{-2t} \)

\[ P(X < 4) = P_X(0) + P_X(1) + P_X(2) + P_X(3) = e^{-2t} \left( 1 + 2t + 2t^2 + \frac{4}{3}t^3 \right) \]

\[ P(X > 1) = 1 - P_X(0) - e^{-2t} \cdot 2t = 1 - e^{-2t} (1 + 2t) \]

3. The joint PMF is the product shown on this line, where the two multiplications are computed below:

\[ P_{N,X}(n, x) = P_N(n) P_X(n, x) \]

where \[ P_N(n) = \binom{t}{n} p^n (1-p)^{t-n}, \quad n = 0, 1, \ldots, t \]

\[ P_X(n, x) = P(\text{all the failures occurred in locations } x+1, \ldots, t \mid n \text{ failures}) \]

We must consider the cases \( n = 0, 1, \ldots, t \). If \( n = 0 \) failures, then

\[ P_X(n, x) = \begin{cases} 1 & \text{if } x = t \\ 0 & \text{otherwise} \end{cases} \]

If \( n = 1, 2, \ldots, t \), then all the failures occur in locations \( x+1, \ldots, t \), and each such configuration arises with the same probability \( \binom{t-x}{n} \)

We obtain

\[ P_X(n, x) = \begin{cases} \binom{t-x}{n} (\frac{t}{n})^n, & x = 0, 1, \ldots, t-n \\ 0 & \text{otherwise} \end{cases} \]
4. \[ P_X | A_n (k|A_n) = \frac{P \{ X=k \} \cap \{ X+Y=n \}}{P (A_n)} = \frac{P \{ (X=k) \cap (X+Y=n) \}}{P (X+Y=n)} = \frac{P \{ (X=k) \} \cdot P(Y=n-k)}{\sum_{i=1}^{n-k} P_X(i) \cdot P_Y(n-i)} \]

- Of course, \( k \) can be any one of the numbers \( \{1, 2, ..., n-1\} \)

- We obtain

\[ P_X | A_n (k|A_n) = \begin{cases} \frac{\binom{n-k-1}{k-1} \cdot \binom{n-k-1}{n-k-i}}{\sum_{i=1}^{n-k} \binom{n-k-1}{i} \cdot \binom{n-k-1}{n-k-i}} & \text{if } k=1, 2, ..., n-1 \\ 0 & \text{for all other } k \end{cases} \]

5. We have \( \sin (Xn) = 0 \) for \( X \) any integer \( 0 \leq X < n \).

\[ \cos (Xn) = (-1)^n \text{ if } X=n. \]

We obtain

\[ E \left[ \sin (Xn) \right] = 0; \quad E \left[ \cos (Xn) \right] = \sum_{n \text{ even}} P_X(n) - \sum_{n \text{ odd}} P_X(n) \]

6. Let \( X \) be the RV equal to the 1st roll such that among the outcome of this and all the previous rolls each of the 6 numbers shows at least once.

Let \( A_i^{(n)} \) be the event that number \( i \) did not show in the first \( n \) rolls.

\[ P(A_i^{(n)}) = \left( \frac{5}{6} \right)^n \]

\[ P(X>n) = P(A_1^{(n)} \cup A_2^{(n)} \cup ... A_6^{(n)}) = \sum_{j=1}^{5} (-1)^j P(A_{i_1} ... A_{i_j}) \]

where the sum is on \( j=1, ..., 5 \) and all the combinations \( i_1, ..., i_j \).

We have

\[ P(X>n) = \left( \frac{5}{6} \right)^n - \left( \frac{4}{6} \right)^n + \left( \frac{3}{6} \right)^n - \left( \frac{2}{6} \right)^n + \left( \frac{1}{6} \right)^n \left( \sum_{k=1}^{n} \right) \]

\[ P_X(n) = -P(X>n) + P(X>n-1) = \left( \frac{5}{6} \right)^n - 5 \left( \frac{4}{6} \right)^n + 10 \left( \frac{3}{6} \right)^n - 10 \left( \frac{2}{6} \right)^n + 5 \left( \frac{1}{6} \right)^n \]

\[ \text{for } n = 6, 7, ... \]

We have \( \frac{147}{10} \) for \( n=6 \).

\[ P_X(n) = 0 \text{ for } n=1, 2, 3, 4, 5 \]

\[ \mathbb{E}X = \sum_{n=6}^{\infty} n P_X(n) = 14.7 \]

\[ \sum_{n=6}^{\infty} n x^{n-1} = \frac{6x^5(1-x)+x^6}{(1-x)^2} \]

\[ \sum_{n=6}^{\infty} n x^{n-1} = \frac{6x^5(1-x)+x^6}{(1-x)^2} \]

\[ n=6 \]
1. \[ f_x(x) = \frac{4}{\pi}, \quad 0 \leq x \leq \frac{\pi}{4} \]

\[
E[\cos(2x)] = \int_0^{\pi/4} \cos 2x \cdot f_x(x) \, dx = \frac{4}{\pi} \int_0^{\pi/4} \cos 2x \, dx = \frac{4}{\pi} \cdot \frac{1}{2} \int_0^{\pi/4} \cos t \, dt = \frac{2}{\pi} \sin t \bigg|_0^{\pi/4} = \frac{2}{\pi}
\]

\[
E[\cos^2(x)] = \frac{4}{\pi} \int_0^{\pi/4} \cos^2 x \, dx = \frac{4}{\pi} \int_0^{\pi/4} \left( \frac{1}{2} + \frac{\cos 2x}{2} \right) \, dx = \frac{1}{\pi} + \frac{1}{\pi}
\]

2. \[ F_x(x) = \begin{cases} 
0 & -\infty < x < 0 \\
1 - \frac{16}{x^4} & 0 < x < \infty 
\end{cases} 
\]

\[ f_x(x) = \frac{d}{dx} F_x(x) = \begin{cases} 
0 & -\infty < x < 0 \\
\frac{32}{x^3} & 0 < x < \infty 
\end{cases} 
\]

\[ E[X] = \int_0^\infty \frac{32}{x^2} \, dx = 32 \cdot \left( \frac{1}{x} \right) \Bigg|_0^\infty = 8 
\]

\[ Var(X) = \int_0^\infty \frac{d^2}{dx^2} F_x(x) \, dx = 32 \ln x \bigg|_0^\infty = \infty \quad (\text{the integral diverges; Var} X \text{ does not exist})
\]

3. \[ E[\ln X] = \int \frac{2 \ln x \, dx}{x^2} = - \frac{2 \ln x}{x} \bigg|_1^2 + \int \frac{2 \, dx}{x^2} = -1 + \ln 2
\]

4. We find \[ E[X] = \int_0^1 (6x^2 - 6x^3) \, dx = \frac{1}{2}; \quad E[X^2] = \int_0^1 (6x^3 - 6x^4) \, dx = \frac{3}{10}; \quad Var X = \frac{1}{20}; \quad \sigma_X = \frac{1}{2 \sqrt{5}}
\]

\[
P(|X - EX| \leq \frac{1}{2 \sqrt{5}}) = \int_{\frac{1}{2} - \frac{1}{2 \sqrt{5}}}^{\frac{1}{2} + \frac{1}{2 \sqrt{5}}} (6x - 6x^2) \, dx = (3x^2 - 2x^3) \bigg|_{\frac{1}{2} - \frac{1}{2 \sqrt{5}}}^{\frac{1}{2} + \frac{1}{2 \sqrt{5}}}
\]

\[
= 3 \left( \left( \frac{1}{2} + \frac{1}{2 \sqrt{5}} \right)^2 - \left( \frac{1}{2} - \frac{1}{2 \sqrt{5}} \right)^2 \right) - 2 \left( \left( \frac{1}{2} + \frac{1}{2 \sqrt{5}} \right)^3 - \left( \frac{1}{2} - \frac{1}{2 \sqrt{5}} \right)^3 \right)
\]

\[
= \frac{3 \cdot 2 \cdot 2}{4 \sqrt{5}} - 2 \cdot 3 \cdot \frac{1}{2 \sqrt{5}} - 2 \cdot \frac{1}{2 \sqrt{5}} \cdot \frac{3}{4 \sqrt{5}} - \frac{2}{8 \sqrt{5}} - \frac{1}{2 \sqrt{5}} \cdot \frac{3}{4 \sqrt{5}} = \frac{1}{\sqrt{5}} \left( 3 - 3 - \frac{1}{3} \right) = \frac{1}{3 \sqrt{5}}
\]
5. The PDF of student's arrival at the bus stop is (counting from 8:45am)

\[ f(x) = \begin{cases} 
\frac{1}{60} & 0 < x < 60 \\
0 & \text{otherwise}
\end{cases} \]

To wait \( \leq 10 \) min he needs to arrive between 8:50 and 9:00 or (9:20, 9:30) to wait \( > 15 \) min he needs to arrive either between 9:00 and 9:30, or between 9:30 and 9:45.

Thus the prob for waiting \( \leq 10 \) min is \( \frac{1}{3} \); for waiting \( > 15 \) min is \( \frac{1}{2} \).

6. We need \( P(b^2 - 4 > 0) \) or \( P(b > 2) \cup \{ b < -2 \} = \frac{1}{3} \).

7. Let \( X \) be a random grade; \( Y = \frac{1}{\sqrt{X}} (X - EX) \)

\[ P(X \geq 90) = P(Y \geq \frac{90 - 72}{\sqrt{7}}) = 1 - \Phi(6.8) \approx 5 \times 10^{-12} \] (this must be a REALLY large class!)

\[ P(Y > 80) \]

\[ P(80 \leq X < 90) = P(3.02 \leq Y < 6.8) = \Phi(6.8) - \Phi(3.02) \approx 0.00126 \]

\[ P(70 \leq X < 80) = P(-0.75 \leq Y < 3.02) = \Phi(3.02) - \Phi(-0.75) \approx 0.772 \]

\[ P(60 \leq X < 70) = P(-4.54 \leq Y < -0.75) = \Phi(-0.75) - \Phi(-4.54) \approx 0.127 \]

\[ P(X \leq 60) = P(Y \leq -4.54) = \Phi(-4.54) \approx 2.81 \times 10^{-6} \]
Problem set 7; solutions

1. \[ F_X(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x 4t^3 dt = x^4 & 0 < x \leq 1 \\ 1 & x > 1 \end{cases} \]

\[ P(Y \leq y) = P(1-3x^2 \leq y) = P[\{X \geq \sqrt{\frac{1}{3}(1-y)}\}] \cup \{X \leq -\sqrt{\frac{1}{3}(1-y)}\} = 1 - F_X(\sqrt{\frac{1}{3}(1-y)}) \]

\[ = \begin{cases} 0 & y \leq -2 \\ 1 - \frac{1}{3}(1-y)^2 & -2 < y \leq 1 \\ 1 & y > 1 \end{cases} \]

\[ f_X(y) = F_X'(y) = \frac{2}{3} (1-y) \text{ if } -2 \leq y \leq 1 \text{ and } f_X(y) = 0 \text{ if } y < -2; y > 1. \]

2. A monitor will last < 15000 hrs if \( X < 1.5 \)

(a) \[ P(X < \frac{1}{3}) = \int_{-\infty}^{1/3} \frac{2}{x^3} dx = -2 \int_{1/3}^{1} \frac{1}{x} dx = \left[-\frac{2}{3}\right]_{1/3} = \frac{2}{3} \]

(b) \[ P(1 < X < 1.25) \setminus \{X < 1.5\} = F_X(1.25) - \frac{2}{3} = \frac{7}{8} - \frac{2}{3} = \frac{3}{8} = \frac{a^2}{2} \]

3. \[ E[e^{ax}] = \int_{-\infty}^{\infty} e^{ax} \cdot \frac{x^2}{2\pi} dx = \frac{e^{2/\pi}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \frac{e^{2/\pi}}{\sqrt{2\pi}} \cdot \sqrt{2\pi} = e^{2/\pi} \]

(The last integral = 1 because \( e^{-x^2/2}/\sqrt{2\pi} \) is a PDF of \( N(a,1) \))

To find the PDF of \( Y \), take \( t > 0 \), then \[ P(Y \leq t) = P(-\sqrt{t} \leq X \leq \sqrt{t}) = 2 \Phi(\sqrt{t}) - 1 \]

\[ f_Y(y) = \frac{d}{dy} (2 \Phi(\sqrt{y}) - 1) = 2 \frac{d}{dy} \int_{-\infty}^{\sqrt{y}} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi y}} e^{-y/2} \]

for \( y > 0 \) and 0 for \( y < 0 \).

4. First, compute \( a \): \[ 1 = a \left( \frac{9}{2} - \frac{1}{2} \right) = 4a \Rightarrow a = \frac{1}{4} \]

Next, \[ E[X] = \int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{x^2}{2} dx = \frac{1}{12} \left( \frac{1}{4} \right)^3 = \frac{1}{6} \]

\[ F_X(x) = \begin{cases} 0 & 1 \leq x \leq 3 \\ \frac{x^2}{8} & 0 \leq x \leq 1 \end{cases} \]

\[ P(B) = P(1.7 \leq X \leq 2.7) = F_X(2.7) - F_X(1.7) = \frac{2}{13} \left( 2.7^2 - 1.7^2 \right) = \frac{1}{8} \left( 7.29 - 2.89 \right) = \frac{4.4}{8} = 0.55 \]

\[ E[X|B] = \int_{-\infty}^{\infty} \frac{f_X(x)}{P(B)} dx \]

\[ E[X|B] = \frac{1}{2.2} \int_{-\infty}^{2.7} \frac{x^2}{2} dx = \frac{1}{6.6} \left( \frac{19.683 - 4.913}{6.6} \right) = 2.2 \]
Problem set 7; solutions (Page 2)

5) We have (by independence) \( f_{XY}(x,y) = f_X(x) f_Y(y) = 1 \cdot 2 e^{-2y} = 2 e^{-2y} \)

\[
P(Y \geq x) = \int_0^\infty \int_x^\infty f_{XY}(x,y) \, dy \, dx = \int_0^\infty \int_x^\infty 2 e^{-2y} \, dy \, dx = \int_x^\infty 2 \, dx = \frac{1}{2} \left( 1 - e^{-2} \right)
\]

Let us find PDF \( Z = X + Y \). If \( Y = y \), then \( Z = X + y \), i.e. \( f_{Z|Y}(z|y) \)

\[
D) \ f_{Z|Y}(z|y) = f_X(z-y) = \begin{cases} 1 & \text{for } y \leq z \leq y+1 \\ 0 & \text{otherwise} \end{cases}
\]

Now let \( Z = 3 \), then \( Y \) can be only between 2 and 3.

\[
f_{Y|Z}(y|3) = \frac{f_{YZ}(y,3)}{f_Z(3)} = \frac{f_{Z|Y}(3|y) f_Y(y)}{f_Z(3)}
\]

The PDF \( f_{Z|Y} \) is computed above in (D).

\[
f_Z(3) = \int_2^3 f_{Z|Y}(3|y) f_Y(y) = \int_2^3 2 e^{-2y} dy = e^{-y} e^{-6}
\]

Finally \( f_{Y|Z}(y|3) = \begin{cases} \frac{2 e^{-2y}}{e^{-y} e^{-6}} & 2 \leq y \leq 3 \\ 0 & \text{otherwise} \end{cases} \)

6) \( X + Y \geq 2 \)

\[
P(X+Y \geq 2) = \int_2^\infty \int_0^{\min(2-x,0)} \lambda_1 e^{-\lambda x} \lambda_2 e^{-\lambda_2 y} \, dy \, dx
\]

\[
= \int_2^\infty \lambda_1 e^{-\lambda x} \left( \int_0^{\min(2-x,0)} \lambda_2 e^{-\lambda_2 y} \, dy \right) \, dx
\]

\[
= \int_2^\infty \lambda_1 e^{-\lambda x} \left( \frac{1}{\lambda_1 + \lambda_2} \right) \left( 1 - e^{-\lambda_1 (2-x)} \right) \, dx
\]

\[
P(\max(X,Y) \geq 2) = 1 - P(\{X \leq 2\} \land \{Y \leq 2\}) = 1 - P(X \leq 2) P(Y \leq 2)
\]

\[
= 1 - (1 - e^{-2\lambda_1})(1 - e^{-2\lambda_2}) = e^{-2\lambda_1} + e^{-2\lambda_2} - e^{-2(\lambda_1 + \lambda_2)}
\]
Problem Set 7, Solution of problem 7

(b) \( X \sim \text{Laplace}(\alpha) \), \( f_X(x) = \frac{1}{\alpha} e^{-\frac{|x|}{\alpha}} \), \( -\infty < x < \infty \)

\( Y \sim \text{Uniform}[0,2] \), \( f_Y(y) = \frac{1}{2} \), \( 0 \leq y \leq 2 \)

\( P(Z > 2 | Y) = P(Z > 2Y) = P(X + Y > 2Y) = P(X > Y) \)

\( = \frac{2}{2} \int_0^2 \int_{y}^{2y} \frac{1}{\alpha} e^{-\frac{x}{\alpha}} dx \ dy = \frac{2}{\alpha} \int_0^2 \left( 1 - e^{-2y} \right) dy = \frac{2}{\alpha} \left( 2 - \frac{1}{2} \right) \approx 0.125 \)

\( \therefore f_Z(z) = \left( f_Y * f_X \right)[z] = \int f_Y(x) f_X(z-x) dx \)

The integrand depends on the sign of \( z-x \):

- If \( z-x < 0 \) or \( x > z \), then \( f_X(z-x) = e^{-z} \)

Then

\[
\int_{-\infty}^0 \frac{1}{\alpha} e^{-\frac{(z-x)}{\alpha}} dx = \frac{1}{\alpha} e^{-\frac{2(z-x)}{\alpha}} \bigg|_0^z = \frac{1}{\alpha} (e^{2z} - e^{-z(2-z)}) , \quad z > 0
\]

\[
\int_{-\infty}^0 \frac{1}{\alpha} e^{-\frac{2(z-x)}{\alpha}} dx = -\frac{1}{\alpha} e^{-2z} + \frac{e^{2z}}{2} = \frac{1}{\alpha} (e^{2z} - e^{-z(2-z)}) , \quad 0 < z < 2
\]

\[
\int_0^2 \frac{1}{\alpha} e^{-\frac{(z-x)}{\alpha}} dx = \frac{1}{\alpha} e^{-\frac{2(z-x)}{\alpha}} \bigg|_0^z = \frac{1}{\alpha} (e^{2z} - e^{-z(2-z)}) , \quad z > 2
\]
Problem set 8, Solutions

1) Let \( X_i \sim \text{geom}( \frac{n-i+1}{n} ) \), \( i = 1, 2, \ldots n \).

\( X_i \) is the RV that is 1 if the first toss lands the marble into an empty box.

Of course, \( P(X_i = 1) = 1 \).

Now there are \( n-1 \) empty boxes left, and if \( X_2 \) is the RV that is 1 if the second, or third, \ldots toss lands the marble into an empty box, then

\[ P(X_2 = 1) = \left( \frac{1}{n} \right)^{n-1} \frac{n-1}{n}, \quad j \geq 1. \]

Generally, \( X_i \) is the RV equal to the number of the toss that takes out one more empty box, counting from the toss that fished resulted in \( i-1 \) nonempty boxes.

We are seeking the quantity

\[ E(X_1 + X_2 + \ldots + X_n) = 1 + \frac{n}{n-1} + \frac{n}{n-2} + \ldots + \frac{n}{2} + n \]

2) We have

\[ \frac{2}{3} = \rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{1}{3} \text{cov}(X, Y) \]

\[ \text{cov}(X, Y) = 2 \]

\[ \text{Var}(3X - 5Y + 7) = 9 \text{Var}(X) + 25 \text{Var}(Y) - 15 \text{cov}(X, Y) = 204 \]
Let $L$ be the random number of bits per letter.

\[ p_L(n) = (1-p)^{n-1}p, \quad n=1,2,3,\ldots \]

\[ P(L > n) = \sum_{\ell=n+1}^{\infty} P(1-p)^{\ell-1} = (1-p)^n, \quad n = 0, 1, 2, \ldots \]

\[ F_T(t) = P(T \leq t) = P(L \leq 1000t) = 1 - P(L > 1000t) = 1 - (1-p)^{1000t} \]

\[ = 1 - e^{-t(-1000\ln(1-p))}, \quad t \geq 0 \]

Answer: $T \sim \exp \left(1000\ln \left(\frac{1}{1-p}\right)\right)$.

For $X = \alpha$ we have $X \sim \text{unif}[0,\alpha]$, so

\[ f_{Y \mid X}(y \mid x) = \begin{cases} \frac{1}{\alpha} & 0 \leq y < x \\ 0 & 1 > y \geq x \geq 0 \end{cases} \]

Since $f_X(x) = 1$, $0 < \alpha < 1$, we obtain

\[ f_{X \mid Y}(x \mid y) = f_{Y \mid X}(y \mid x) f_X(x) = \begin{cases} \frac{y}{\alpha x} & 0 \leq y < x, \ 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ f_Y(y) = \int_0^{\min(1, y/\alpha)} \frac{dy}{x} = -\ln y \text{ for } 0 < y \leq 1 ; \quad EY = -\int_0^1 y \ln y \, dy = -\frac{y^2}{2} \ln y \bigg|_0^1 + \int_0^1 y \, dy = \frac{1}{2} \]

Using law of iterated expectations we obtain

\[ EY = E_X [E[Y \mid X]] = E \left( \frac{X}{2} \right) = \frac{1}{4} \]

Area: $E[XY] = \int_0^1 \int_0^x \frac{1}{2} \, dy \, dx = \int_0^1 \frac{1}{2} x^2 \, dx = \frac{1}{6}$
Problem Set 8, Solutions (page 3)

5) Since \( M_X(t) = \frac{e^t - 1}{t} \quad (t \neq 0) \)

\[
M_Y(s) = E\left[ e^{s(aX+b)} \right] = e^{bs} E e^{asX} = e^{bs} E M_X(as)
\]

\[
= e^{bs} \frac{e^{as} - 1}{as} = \frac{e^{(a+b)s} - e^{bs}}{(a+b)s - bs}
\]

We recognize \( M_Y(s) \) as the transform of a uniform RV on \([b, a+b]\)

6) We have \( M_X(s) = \left( \frac{e^s}{3} + \frac{2}{3} \right)^4 \)

so \( X \sim \text{binom} \left(n=4, p=\frac{1}{3}\right) \)

and \( P(X \leq 2) = \sum_{i=0}^{2} \binom{4}{i} \left(\frac{1}{3}\right)^i \left(\frac{2}{3}\right)^{4-i} = \left(\frac{2}{3}\right)^4 + 4 \cdot \frac{1}{3} \cdot \left(\frac{2}{3}\right)^3 + 6 \cdot \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2 \)

\[
= \frac{16}{81} + \frac{32}{81} + \frac{24}{81} = \frac{72}{81} = \frac{8}{9}/
\]