(i) (8 pts.)
Consider a system with input $x(t)$ and output $y(t)$ related as:
\[ y(t) = x(t^2) + x(3t) \]
Is this system:
(a) (2 pts.) Linear?
(b) (2 pts.) Time-invariant?
(c) (2 pts.) Causal?
(d) (2 pts.) Bounded-input bounded-output (BIBO) stable?

(ii) (6 pts.)
Consider the linear time-invariant (LTI) system with input $x[n]$ and output $y[n]$ defined by the following difference equation:
\[ y[n] - \frac{1}{3}y[n - 1] = x[n] \]
Find the output when the input is:
(a) (3 pts.) $x[n] = \left(\frac{1}{2}\right)^n u[n]$, where $u[n]$ is the unit step function.
(b) (3 pts.) $x[n] = \sin\left(\frac{\pi}{2} n\right)$.

(iii) (6 pts.)
Determine if the signals $y_1(t)$, $y_2[n]$ and $y_3(t)$ below are periodic.

(a) (2 pts.) $y_1(t)$ is the output of the system described by:
\[ y(t) = \int_{t-5}^{t} x(\tau)d\tau \]
to the input $x_1(t) = \cos(2t)$.
(b) (2 pts.) $y_2[n]$ is the output of the system described by:
\[ y[n] = \sum_{k=n-5}^{n} x[k] \]
to the input $x_2[n] = \cos[2n]$.
(c) (2 pts.) $y_3(t)$ is the output of the system described by:
\[ y(t) = x^2(t) \]
to the input $x_3(t) = \sin(\cos(4t))$. 
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(i)

(a) Yes, linear, because, when input is \( a_1 x_1(t) + a_2 x_2(t) \), the output is \( a_1 y_1(t) + a_2 y_2(t) \).

(b) No, not time-invariant, because, when input is \( x(t-t_0) \), the output is not \( y(t-t_0) \).

(c) No, not causal, because, e.g., \( y(1) = x(1)+x(3) \), i.e., output at time 1 depends on input at time 3.

(d) Yes, BIBO stable, because if \( |x(t)| < A \) for all \( t \), then \( |y(t)| < 2A \) for all \( t \).

(ii)

(a) \( H(e^{jw}) = 1/(1-1/3*e^{-jw}) \), \( X(e^{jw}) = 1/(1-1/2*e^{-jw}) \).

Then, \( Y(e^{jw}) = 1/(1-1/3*e^{-jw})*1/(1-1/2*e^{-jw}) = -2/(1-1/3*e^{-jw}) + 3/(1-1/2*e^{-jw}) \).

And, \( y[n] = -2(1/3)^n u[n] + 3(1/2)^n u[n] \).

(b) \( x[n] = (e^{jn \pi/2}) - e^{-jn \pi/2})/2j \).

Then, \( y[n] = (H(e^{jn \pi/2}) e^{jn \pi/2}) - H(e^{-jn \pi/2}) e^{-jn \pi/2})/2j \)

And, \( y[n] = 9/10(sin(pi/2 n) - 1/3cos(pi/2 n)) \).

(iii)

(a) Yes, \( y_1(t) \) is periodic, because \( x_1(t) \) is periodic and it goes through an LTI to yield \( y_1(t) \).

(b) No, \( y_2[n] \) is not periodic, because \( x_2[n] \) is not periodic.

(c) Yes, \( y_3(t) \) is periodic, because \( x_3(t) \) is periodic and we obtain \( y_3(t) \) by squaring \( x_3(t) \).