We begin by speaking of an abstract group $G$ group acting on a set $S$. We denote by $\text{Aut}(S)$ the collection of automorphisms of $S$, i.e., one-to-one, onto maps $S \to S$.

**Definition 1.** A left action of a group $G$ on a set $S$ is a map $\Phi : G \times S \to S$ such that

(i) $\forall x \in S \quad \Phi(e, x) = x$ where $e$ is the identity element of $G$

(ii) for every $g, h \in G$

$$\Phi(g, \Phi(h, x)) = \Phi(gh, x) \quad \forall x \in S.$$ 

**Remark 2.** It follows from (i) and (ii) above that

$$\Phi_g \circ \Phi_{g^{-1}} = \Phi_{g^{-1}} \circ \Phi_g = 1_S \quad \text{where}$$

$$\Phi_g : S \to S \quad \text{is given by} \quad \Phi_g(x) = \Phi(g, x).$$

Thus $\Phi_g$ is an automorphism. If $S$ has additional structure (e.g., manifold), then we can ask that $\Phi_g$ be a diffeomorphism for each $g$. This can be built into the definition (see Lecture 3, Definition 20).
**Definition 3** Let $\mathfrak{G}$ be a group action of $G$ on $S$. For $x \in S$, the **orbit** of $x$ is given by

$$\mathcal{O}_x = G \cdot x \equiv \{ \mathcal{O}_g(x) \mid g \in G \}.$$ 

An action is **transitive** if $\mathcal{O}_x = S$, i.e., there is just one orbit. It is **effective** (or **faithful** ) if $\mathcal{O}_g = \mathcal{O}_e \Rightarrow g = e$, i.e., $g \mapsto \mathcal{O}_g$ is one-to-one. An action is **free** if, for each $x \in S$, $g \mapsto \mathcal{O}_g(x)$ is one-to-one. 

**Definition 4** The relation of belonging to the same orbit is an equivalence relation on $S$. The set $S$ is carved up into **orbits**. Let $S/\mathcal{G}$ be the set of equivalence classes (orbits). Let $\Pi : S \to S/\mathcal{G}$ be defined by

$$\Pi(x) = [x] = \mathcal{O}_x.$$

We call $S/\mathcal{G}$ the **orbit space**.

If $S$ has a topology, one can give $S/\mathcal{G}$ the quotient topology, i.e., $U \subset S/\mathcal{G}$ is open iff $\Pi^{-1}(U)$ is open in $S$.

**Example 6** $G = \mathbb{R}^n$, $S = \mathbb{R}^n$, $A \in \mathbf{Mat}(n)$

$$\mathfrak{G} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

$$(t, x) \mapsto e^{tA}x$$

is a group action (see also def'19 Lecture 3)
If $A$ has no eigenvalues on the imaginary axis then the action is effective. Is it free?

Sophus Lie studied groups as transformation groups acting on sets (e.g., sets of differential equations). He was mainly interested in what we now call Lie algebras of Lie groups.

**Definition 6** A Lie group $G$ is a finite dimensional smooth manifold that is a group for which the group operations of multiplication

$$
\cdot : G \times G \rightarrow G \quad (g_1, g_2) \rightarrow g_1 \cdot g_2
$$

and inversion

$$
\cdot^{-1} : G \rightarrow G \quad g \rightarrow g^{-1}
$$

are smooth. Let $e =$ identity.

We have already seen many examples of Lie groups in our study of manifolds. The most famous Lie groups are the classical Lie groups of invertible linear transformations on a vector space (over $\mathbb{R}$ or $\mathbb{C}$). We list them, somewhat loosely:
Field = $\mathbb{R}$
\[
\text{GL}(n; \mathbb{R}) = \left\{ X : X \text{ is an } n \times n \text{ matrix, } \det(X) \neq 0 \right\}
\]
\[
\text{SL}(n; \mathbb{R}) = \left\{ X : X \in \text{GL}(n; \mathbb{R}), \det(X) = 1 \right\}
\]
\[
\text{O}(n; \mathbb{R}) = \left\{ X : X \in \text{GL}(n; \mathbb{R}), \ X^T X = I \right\}
\]
\[
\text{SO}(n; \mathbb{R}) = \left\{ X : X \in \text{O}(n; \mathbb{R}), \ \det(X) = 1 \right\} = \text{SL}(n; \mathbb{R}) \cap \text{O}(n; \mathbb{R})
\]
\[
\text{Sp}(2n; \mathbb{R}) = \left\{ X : X \in \text{GL}(2n; \mathbb{R}), \ X^T J X = J \right\}
\]
\[\text{here } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \]
\[
\text{O}(p,q; \mathbb{R}) = \left\{ X : X \in \text{GL}(p+q; \mathbb{R}), \ X^T \Sigma_{p,q} X = \Sigma_{p,q} \right\}
\]
\[\text{here } \Sigma_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \]

Field = $\mathbb{C}$ . Denote by $^*$ the Hermitian adjoint/transpose.
\[
\text{GL}(n; \mathbb{C}) = \left\{ X : X \text{ is an } n \times n \text{ matrix over } \mathbb{C} \right\}
\]
\[ X \text{ is invertible} \]
\[
\text{SL}(n; \mathbb{C}) = \left\{ X : X \in \text{GL}(n; \mathbb{C}), \ \det(X) = 1 \right\}
\]
\[
\text{U}(n; \mathbb{C}) = \left\{ X : X \in \text{GL}(n; \mathbb{C}), \ X^* X = I \right\}
\]
\[
\text{SU}(n; \mathbb{C}) = \left\{ X : X \in \text{U}(n; \mathbb{C}), \ \det(X) = 1 \right\} = \text{U}(n; \mathbb{C}) \cap \text{SL}(n; \mathbb{C})
\]
\[
\text{Sp}(2n; \mathbb{C}) = \left\{ X : X \in \text{GL}(2n; \mathbb{C}), \ X^* J X = J \right\}
\( O(b, v); \mathbb{G} \) = \{ \( x \in \mathcal{G}(p+q; \mathbb{C}); x^* x = \mathbb{E} \}_{b, v} \}

These are all groups of homogeneous transformations of the form \( y \mapsto X y \).
One associates groups of inhomogeneous transformations by including translating on the underlying vector space,

\[ y \mapsto X y + \xi. \]

Examples:

(i) \( \text{Aff} (2) = \{ (a, b) : a > 0, b \in \mathbb{R} \} \)

\( \text{(inhomogeneous)} \)
acts via affine transformation on \( \mathbb{R}^2 \)
\[ x \mapsto ax + b. \]
This plays a basic role in the theory of affine wavelets.

(ii) \( \text{SE}(3) = \{ (P, b) : P \in \text{SO}(3), b \in \mathbb{R}^3 \} \)

acts on \( \mathbb{R}^3 \) by rigid motions
\[ x \mapsto Px + b. \]

Note \( \text{SE}(3) = \text{SO}(3) \times \mathbb{R}^3 \)
a semidirect product (see Lecture 3, Example 18 on unicycles for \( \text{SE}(2) \). \( \text{SE}(3) \) arises
in the modeling and analysis of heavy spinning tops and underwater vehicles. It also arises in the study of robotic manipulators (linked multi-rigid body systems).

Remark 8

When we speak of Lie groups acting (on the left) on a smooth manifold we modify definition 1, asking that \( \Phi \) be smooth.

Remark 9

Action on the right means \( \Phi : M \times G \to M \) satisfying \( \Phi_e = \text{Id}_M \); and

\[
\Phi (\Phi (m, h), g) = \Phi (m, h \cdot g)
\]

for \( m, h, g \in G \) and \( m \in M \).

There are a number of key results as to the circumstances under which \( M/G \) is a smooth manifold. We postpone this for later. Below we consider the setting in which \( M = G \). There are two actions:

left translation

\[
L : G \times G \to G,
\]

\[
(\ell, h) \mapsto \ell \cdot h = \ell h \quad L_\ell (h) = \ell \cdot h
\]

right translation

\[
R : G \times G \to G,
\]

\[
(h, g) \mapsto R_g (h) = h \cdot g
\]
Clearly $L$ and $R$ are smooth actions. They are both free actions.

For a diffeomorphism $\varphi : M \to M$ define

$$\varphi^* : \text{Vect}(M) \to \text{Vect}(M)$$

to be (an invertible) operation on vector fields by

$$\varphi^* X = \mathcal{D}\varphi \circ X \circ \varphi^{-1}$$

This operation is called \underline{push-forward}.

Thus

$$(\varphi^* X)(m) = (\mathcal{D}\varphi)(\varphi^{-1}(m)) \cdot (\varphi^{-1}(m))$$

We say that a vector field $X$ is \underline{invariant} under a diffeomorphism $\varphi$ provided

$$\varphi^* X = X,$$

equivalently, $\forall \ m \in M$

$$(\mathcal{D}\varphi)(\varphi^{-1}(m)) = X(m).$$

In that case we also say $\varphi$ is a \underline{symmetry} of $X$. 
For a smooth action $\Phi$ of a (Lie) group $G$ on a manifold $M$, we say that a vector field $X$ on $M$ is invariant under the action $\Phi$ if

$$\Phi_g \ast X = X + g \in G.$$ 

We will now specialize this concept to the case of the actions $L$ (and $R$) of a Lie group $G$ on itself. First a bit of notation.

Let $T^L_g \in (DLg)_g$ be the linearization of $L_g$ at $h \in G$. Similarly for $R_g$. Then a vector field $X$ on $G$ is left-invariant (i.e. under the action of left-translation) if, for $x, y \in G$, $T^L_x X(y) = X(yh) + h \in G$.

Let $\mathfrak{X}_L(G)$ denote the set (vector space) of all left-invariant vector fields on $G$. Define

$$p_1 : \mathfrak{X}_L(G) \rightarrow T_e G$$

$$X \mapsto X(e)$$

and

$$p_2 : T_e G \rightarrow \mathfrak{X}_L(G)$$

$$\xi \mapsto X_\xi$$

defined by...
\[ X_\delta (g) = T_e L_\delta g. \]

Then \( p_1 \circ p_2 = \text{id}_{T_e G} \) (\( \because T_e e = e \))

and \( p_2 \circ p_1 = \text{id}_{\mathfrak{g}_L (G)} \) (by left inv.)

Thus \( \mathfrak{g}_L (G) \) and \( T_e G \) are vector spaces.

**Theorem**

\[ [\mathfrak{g}_L (X), \mathfrak{g}_L (Y)] = \mathfrak{g}_L [X, Y] \]

for any diffeomorphism \( \varphi \)

**Proof**

is left as an exercise. \( \square \)

(see R. Abraham & J. E. Marsden, Foundations of Mechanics 1978)

**Definition**

On \( T_e G \) define a Lie bracket

\[ [\xi, \eta] = [X_\xi, X_\eta] \]

where the Lie bracket on the right is defined as the Jacobi-Lie bracket of vector fields. This makes \( p_2 \) and \( p_1 \) Lie algebra isomorphisms.

**Definition**

The vector space \( \mathfrak{g}_L = T_e G \) with Lie bracket as above is the Lie algebra of \( G \).