Recall that, given a vector space $V$ of dimension $n$, there is another vector space $V^*$ called the linear dual or simply dual:

$$V^* = \{ f : V \to \mathbb{R} \mid f \text{ is linear} \}.$$

The choice of a basis $\{ e_1, e_2, \ldots, e_n \}$ for $V$ determines a dual basis $\{ e^1, e^2, \ldots, e^n \}$ for $V^*$ satisfying

$$e^i(e_j) = \delta^i_j,$$

Kronecker symbol.

For a smooth manifold $M$, let $V = T_m M$ be the tangent space to $M$ at $m$. Let $V^* = T^*_m M$. We call this the cotangent space.

There is also the cotangent bundle,

$$T^* M = \bigcup_{m \in M} T^*_m M,$$

a smooth manifold and associated projection

$$T^* M \xrightarrow{T^*_M} M.$$

We denote tangent vectors at $v_m, w_m, \ldots$ using Roman letters and cotangent vectors as
Definition 2

A covector field $\Theta$ is a cross-section of the bundle $(T^*M, \pi^*: M \rightarrow T^*M)$, i.e., it is a smooth map

$$\Theta : M \rightarrow T^*M$$

such that $\pi^* \circ \Theta = \text{id}_M$.

A covector field $\Theta$ assigns to each $m \in M$, a cotangent vector $\Theta_m \in T^*_m M$.

On a chart $(U, \phi)$ of $M$, $m \in U$, we have constructed a basis for $T^*_m M$, namely the set of basis derivation $D^x_i (m) = 1, 2, \ldots, k$ where $k = \dim (M)$. See Lecture 2 (page 15-17).

Here we intend to construct the dual basis for $T^*_m M$. (Note — as usual $\phi = (x_1, \ldots, x_k)$ where $x_i$ are the coordinate functions on the chart $U$.) Again we will use the notation $(x_1, \ldots, x_k)$ to denote the $k$-tuple of coordinate functions as well as the coordinates of a point in $\phi(U) \subset \mathbb{R}^k$. The context will make clear what we mean.

Suppose $M$ and $N$ are smooth manifolds.

Let $\psi : M \rightarrow N$ be a smooth map, $m \in M$ $\psi(m) \in N$. Let $\gamma : [0, T] \rightarrow M$ be a smooth curve in $M$ such that $\gamma(t) = m$ for some $t \in [0, T]$. 
Then $\psi \circ \xi$ is a smooth curve in $N$ such that $(\psi \circ \xi)(t) = \psi(m)$.

Let $f$ be a partial function on $N$ at $\psi(m)$, i.e., there is an open set $V \subseteq N$ and $\psi(m) \in V$ and $f: V \to \mathbb{R}$ is smooth. Let $U = \psi^{-1}(V)$. $U$ is an open set containing $m$. Consider $\psi^*f$ defined by

$$(\psi^*f)(p) = f(\psi(p)) \quad p \in U.$$

**Definition 3.** We call $\psi^*f$ the pull-back of $f$. It is clearly a partial function on $M$ at $m$.

Now $\dot{x}(t) \in T_M$ is defined by

$$(\dot{x}(t))^\#_m \dot{h} = \frac{d}{dt} h(x(t)), \quad h \in F(M,m).$$

We have already seen that $\dot{x}(t)$ is a derivation.

It maps to a derivation on $F(N, \psi(m))$ as follows. Define $d_m (d\psi) \dot{x}(t)$ by

**Definition 4.**

$$(d_m (d\psi) \dot{x})(t) f = \frac{d}{dt} (f \circ \psi)(\xi(t)), \quad f \in F(N, \psi(m)).$$

Since $\psi \circ \xi$ is a smooth curve in $N$ passing through $\psi(m)$, it follows that $(d_m (d\psi) \dot{x})(t)$ is a tangent vector in $T_M$ and is clearly a derivation (by chain rule).

Observe that

$$\frac{d}{dt} f(\psi \circ \xi)(t) = \frac{d}{dt} (f \circ \psi)(\xi(t)).$$
\[ \frac{d}{dt} \left( \psi^* f \right) (\chi(t)) = \chi(t) \psi^* f \]

We have the nice formula

\[ \left( d\psi \right)_m^{\ast} \chi(t) = \chi(t) \psi^* \]

Letting \( \chi = \Phi^{-1} \circ \tilde{\chi} \) as in Lecture Notes #2 page 16, we see that

\[ \left( d\psi \right)_m^{\ast} \mathbf{D}(m) = \mathbf{D}(m) \psi^* \]

for the basis tangent vectors \( \mathbf{D}_i(m) \), \( i = 1, 2, \ldots, k \). This extends to all tangent vectors \( s \in TM \)

\[ \left( d\psi \right)_m^{\ast} s = s \psi^* . \]

It is clearly linear in \( s \).

Suppose \( N = \mathbb{R}^k \), with global coordinate \( x \). Any tangent vector in \( T\mathbb{R}^k \) can be written as \( \frac{d}{dx} \) for some \( c \in \mathbb{R} \). So we identify

\[ T\mathbb{R}^k = \mathbb{R} \]

Thus given \( \Phi : M \to \mathbb{R}^k \) we have a linear map
\[(d\psi)_m : TM \rightarrow TR \subseteq R\]

Thus \((d\psi)_m \in \mathfrak{X}^* M\). We construct a basis for \(\mathfrak{X}^* M\) dual to the basis \(\{\partial \xi_i^* m \mid i = 1, 2, \ldots, \ell\}\) for \(\mathfrak{X}_m M\).

Let \(\psi = x_i : M \rightarrow R\)

\[m \mapsto x_i(m) \text{ the } i\text{th coordinate of } m \text{ in the chart } (U, \phi)\]

for \(f : R \rightarrow R\),

\[
(d\psi)_m \partial \xi_j^* m = \partial \xi_j^* (m) \quad (Lecture 2, page 16)
\]

\[
= \partial \left((x_i^* f) \circ \phi^{-1}\right) \quad \text{definition of pull-back}
\]

\[
= \partial \left((f \circ x_i) \circ \phi^{-1}\right) \quad \text{composition}
\]

\[= \begin{cases} 
0 & \text{if } i \neq j \\
\frac{df(x_i)}{dx_i} & \text{if } i = j
\end{cases}
\]
With the identification of $T^*\mathbb{R}^n$ with $T^*_{x_i(m)}\mathbb{R}^n$ as in page (4), it follows that

$$\langle dx_i \rangle_m \begin{pmatrix} \frac{\partial}{\partial x_j} \end{pmatrix}_{(m)} = \delta_i^j$$

the Kronecker symbol

Thus $\{dx_i\}_{m}$ is the dual basis to $T^*_{m}\mathbb{R}^n$ we are looking for.

Any $\Theta_m \in T^*_{m}\mathbb{R}^n$ can be written as a linear combination

$$\Theta_m = \sum_{i=1}^{k} \Theta_{mi} \langle dx_i \rangle_m$$

where the components $\Theta_{mi}$ are real numbers.

Now as we vary $m$ all over $M$, we get a covector field $\Theta$, expressed in local coordinates as

$$\Theta = \sum_{j=1}^{k} \Theta_i \frac{\partial}{\partial x_i}$$

where $\Theta_i$ are smooth functions on (a chart of) $M$, satisfying $\Theta_i = \Theta(D_{x_i})$, the covector field $\Theta$ evaluated on a basis vector field $D_{x_i}$.

Compare this with the Basis Representation of a vector field (page 2 of Lecture Notes #3).
Given a covector field $\Theta$ and a vector field $X$ we have the **evaluation** $\Theta(X)$, a function on $M$, expressed locally by

$$\Theta(X) = \left( \sum_{i=1}^{k} \Theta_i \cdot dx_i \right) \left( \sum_{j=1}^{k} X_j \cdot dy_j \right)$$

$$= \sum_{i=1}^{k} \Theta_i X^i$$

$$= \text{sum of products of component functions}$$

**Definition 5** A differential form of degree 1 is a covector field (**short-hand terminology: 1-form**)

All differential forms of degree 1 form a vector space $\Omega^1(M)$. It is infinite dimensional in general. Given any $\Theta \in \Omega^1(M)$ and $f \in C^\infty(M)$ $f \Theta \in \Omega^1(M)$ is defined by point-wise multiplication

$$(f \Theta)_m = f(m) \Theta_m \quad m \in M.$$

One way to generate a 1-form (i.e. differential form of degree 1, or covector field) is to start with a **smooth function** $\psi$ and apply $d$ to it. Observe:

$$\psi : M \to \mathbb{R}$$

$$\left( d\psi \right)_m : T^*M \to T\mathbb{R} \subset \mathbb{R}$$
Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be any smooth function.

$$(d \psi)_m \cdot D_{x_i}(m) \cdot f$$

$$= D_{x_i}(m) \cdot f$$

(page 3 and 4)

$$= \frac{\partial}{\partial x_i} \left( (\psi \circ f) \circ \phi^{-1} \right)$$

(page 16, lecture 2)

$$= \frac{\partial}{\partial x_i} \left( f \circ (\psi \circ \phi^{-1}) \right)$$

(-definition of pullback)

$$= \frac{\partial}{\partial x_i} \left( f \circ (\psi \circ \phi^{-1}) \right)$$

(associativity of composition)

$$= \frac{\partial}{\partial x_i} \left( \psi \circ \phi^{-1} \right) \cdot df \bigg|_{x_i \phi^{-1}(x)}$$

(-chain rule)

$$\Rightarrow (d \psi)_m \cdot D_{x_i}(m) = \frac{\partial}{\partial x_i} \left( \psi \circ \phi^{-1} \right) \cdot df \bigg|_{x_i \phi^{-1}(x)} \psi \circ \phi^{-1}(x)$$

By the identification of $T_{1R} \cong R$, we write the component of $(d \psi)_m$ in the direction $d_i (dx_i)_m$ by the evaluation

$$(d \psi)_m \cdot D_{x_i}(m) = \frac{\partial}{\partial x_i} \left( \psi \circ \phi^{-1} \right)$$
Thus we expand

\[(dy)_m = \sum_{i=1}^{k} \frac{2}{\partial x_i} \cdot (dx_i)_m\]

as we vary m, we have a smooth 1-form expressed in local coordinates as

\[dy = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} (\psi \circ \phi^{-1}) \cdot dx_i\]

\[= \sum_{i=1}^{k} 2 \psi \cdot dx_i\]

where \(\tilde{\psi}\) is \(\psi\) expressed in local coordinates. It is customary to drop the "tilde" altogether.

**Example 6**

Let \(M = \mathbb{R}^2\)

Let \(\psi : \mathbb{R}^2 \to \mathbb{R}\)

\[(x, y) \mapsto \frac{1}{2} (x^2 + y^2)\]

Then \(d\psi \in \omega^1(\mathbb{R}^2)\) is the 1-form expressed in (global) coordinates

\[dy = x \, dx + y \, dy\]

**Example 7**

Let \(M = \mathbb{R}^2\). Consider the 1-form

\[\Theta\] expressed in (global) coordinates as

\[\Theta = \theta_1 \, dx + \theta_2 \, dy\]

Is there a function \(\psi : \mathbb{R}^2 \to \mathbb{R}\) such that...
\[ d\psi = 0? \]

From the discussions above, it is clear that, for this to be true, the partial differential equations
\[ \theta_1 = \frac{\partial \psi}{\partial x} \]
\[ \theta_2 = \frac{\partial \psi}{\partial y} \]

hold for some function \( \psi \). A necessary condition for this to be true is that
\[ \frac{\partial \theta_1}{\partial y} = \frac{\partial \theta_2}{\partial x} \]

For \( \theta_1 = x^2 y \); \( \theta_2 = xy^2 \) this condition fails. For \( \theta_1 = xy^2 \); \( \theta_2 = x^2 y \) this condition holds.

Thus we see that not all 1-forms come from functions.

**Definition 8** We call \( \Omega^0(M) \) the space of differential forms of degree 0 or plainly 0-forms. This is defined to be \( \Omega^0(M) = C(M) \) the space of smooth functions.

The process in pages 8-9 tells us how to define \( d : \Omega^0(M) \to \Omega^1(M) \). Constant functions go to the trivial 1-form under this map.

**Definition 9** If \( \theta \in \Omega^1(M) \) satisfies \( \theta = d\psi \) globally for some \( \psi \in \Omega^0(M) \), we say \( \theta \) is exact.