Let \( M \) be a smooth manifold and let \( TM \) denote the associated tangent bundle, also a smooth manifold, with projection \( \pi : TM \rightarrow M \).

**Definition 1**
\[ \pi_m(v_m) = m \text{ for each } v_m \in TM. \] A vector field \( X \) is a cross-section of the tangent bundle, i.e., it is a smooth map \( X : M \rightarrow TM \) that assigns to each \( m \in M \), a particular tangent vector \( X_m \in TM \). The term cross-section derives from \( \pi \circ X = \text{id}_M \), the identity map on \( M \).

Let \( (U, \phi) \) be a chart at \( m \in M \), i.e., \( U \subset M \), \( \phi : U \rightarrow \mathbb{R}^k \) be a homeomorphism onto its image and \( m \in U \). Denote
\[ \phi = (x_1, x_2, \ldots, x_k) \] where \( x_i \) are coordinate functions on \( U \). Then, we have defined \( D_{x_i}^m(f) \) to be the tangent vector at \( m \) satisfying
\[ D_{x_i}^m(f) = \frac{\partial}{\partial x_i} \left( \phi \circ f \circ \phi^{-1}(\phi(m)) \right) \]
for \( f \in C^\infty(M) \). See Lecture 2, page 16.

The tangent vectors \( D_{x_i}^m \) form a basis for \( TM \) and we can write, for a particular \( m \in M \),
\[ X_m = \sum_{i=1}^{k} c_i \cdot D_{\xi_i}(m) \]

Suppose \( p \in U \), \( p \) not necessarily \( m \). Then \( D_{\xi_i}(p) \) satisfies

\[ D_{\xi_i}(p) f = \frac{\partial (f \circ \phi^{-1})}{\partial x_i} \cdot \phi(p) \]

and one can define \( X_p \) in terms of \( D_{\xi_i}(p) \), as we did \( X_m \). Letting \( p \) run over all of \( U \), we have

\[ X(p) \equiv X_p = \sum_{i=1}^{k} f_i(p) \cdot D_{\xi_i}(p) \]

where \( f_i \) are suitable smooth functions on \( U \).

In fact, from \( D_{\xi_i}(p) x_j = \delta_{ij} \) the Kronecker delta (Lecture 2, page 17(c)), and \( x_i \in \mathcal{F}(M, p) \) \( \forall p \in U \), it follows that

\[ f_i(p) = (X x_i)(p) \equiv X^i \]

Basis rep 2. We can write

\[ X = \sum_{i=1}^{k} (X x_i) \cdot D_{\xi_i} = \sum_{i=1}^{k} \sum_{i=1}^{k} x_i D_{\xi_i} \]

where \( D_{\xi_i} \) are basis vector fields.

This mimics Theorem 15 (Lecture 2, page 17(a)).

Often, the definition of vector field can be applied in a local manner, i.e. \( X : U \subseteq M \rightarrow TM \) in a local coordinate system defined on an open set \( U \subseteq M \), \( (\text{id})_{TM} \cdot X = \text{id} \). Everything we have done above makes sense in the local setting as well.
A vector field acts on a smooth function to produce a new smooth function:

$$f \mapsto Xf.$$ 

We define $Xf$ by

$$(Xf)(p) = X_p f.$$ 

From the derivation properties of tangent vectors it follows that, for functions $f, g$,

$$X(fg)(p) = X_p (fg)$$

$$= g(p) X_p f + f(p) X_p g$$

$$= g(p) (Xf)(p) + f(p) (Xg)(p).$$

Hence $X(fg) = g Xf + f Xg$. It is also clear that $X(af + bg) = a Xf + b Xg$ for $a, b \in \mathbb{R}$.

It is possible to obtain a new vector field from old by multiplication by a function: given $X$ a vector field and $f$ a function, define the vector field $fX$ by

$$(fX)_p = f(p) X_p, \quad p \in M.$$ 

Similarly $X + Y$ is defined by $(X + Y)_p = X_p + Y_p$.

We have shown that the collection of all vector fields on $M$ is a module over the ring of smooth functions with a derivation property.
What other ways to generate new vector fields from old?

Consider \( XY \) defined by \((XY)f = X(Yf)\) for any smooth \( f \) on \( M \). Is \( XY \) a vector field?

We compute
\[
(XY)(af + bg) = X(Y(af + bg)) = X(aYf + bYg) = a(XY)f + b(XY)g
\]
thus linearity of \( XY \) on the ring of smooth functions holds.

Next,
\[
XY(fg) = X(Y(fg)) = X(fYg + gYf) \quad \text{by derivation property of } Y
\]
\[
= (XF)(Yg) + f X(Yg) + (Xg)(Yf) + g X(Yf) \quad \text{derivative property of } f
\]
\[
\ne fXY(g) + g XY(f)
\]

Hence \( XY \) is not a derivation. Hence it cannot be a vector field.

But \( XY - YX \) satisfies
\[
(XY - YX)(fg) = f(XY - YX)(g) + g(XY - YX)(f)
\]

from the above calculation. Hence \( XY - YX \) is a derivation on the ring of smooth functions. Hence it can be a vector field. Which one?

Let \( X = \sum_{i=1}^{k} X^i \partial / \partial x_i \); \( Y = \sum_{i=0}^{l} Y^i \partial / \partial x_i \) in local coordinates.
5. \[ \text{Then, } (XY - YX)f = X(Yf) - Y(Xf) \]

\[ = \sum_{i=1}^{k} x^i D_{x^i} \left( \sum_{j=1}^{k} y^j D_{x^j} f \right) \]

\[ - \sum_{i=1}^{k} y^i D_{x^i} \left( \sum_{j=1}^{k} x^j D_{x^j} f \right) \]

\[ = \sum_{i, j=1}^{k} \left( x^i D_{x^i} y^j - y^i D_{x^i} x^j \right) D_f \]

(applying derivative properties of basis vectors fields, etc.)

**Definition 3**

Then we have expressed the commutator

\[ (XY - YX) = \sum_{i, j} \left( x^i D_{x^i} y^j - y^i D_{x^i} x^j \right) D_f \]

Indeed a vector field. We introduce a new notation

\[ [X, Y] = XY - YX, \text{ called the Jacobi-Lie bracket.} \]

In this notation, the vector field \([X, Y]\) has the components

\[ [X, Y]^j = \sum_{i=1}^{k} \left( x^i D_{x^i} y^j - y^i D_{x^i} x^j \right) \]

(i) Linearity
\[ [aX + bY, Z] = a[X, Z] + b[Y, Z] \]
\[ \forall a, b \in \mathbb{R}. \]

(ii) \[ [X, Y] = -[Y, X] \]

(iii) Jacobi identity
\[ [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \]
Whenever \( X, Y, Z \) are vector fields.

**Proof of (iii)**

\[ [X, [Y, Z]](f) = X(Y(Zf)) - Y(X(Zf)) = X(Y(Zf) - Z(Yf)) - (YZ - ZY)(Xf) \]
\[ = X(Y(Zf)) - X(Z(Yf)) - Y(Z(Xf)) + Z(Yf) \]

\[ [Y, [Z, X]](f) \]
\[ = Y(Z(Xf)) - Y(X(Zf)) - Z(X(Yf)) + X(Z(Yf)) \]

\[ [Z, [X, Y]](f) \]
\[ = Z(X(Yf)) - Z(Y(Xf)) - X(Y(Zf)) + Y(Zf) \]

Add and verify that sum of r.h.s is \( 0 \) for \( \forall f \).

[Note less painful than using coordinates!]
Example 5. Let \( M = \mathbb{R}^2 \). There is a global chart. Any smooth vector field \( \mathbf{X} = X^1 \frac{\partial}{\partial x^1} \) where \( X^1: \mathbb{R}^2 \to \mathbb{R} \) is a smooth function. The collection of all smooth vector fields on \( \mathbb{R}^2 \) is in a 1-to-1 correspondence with the set of all smooth scalar functions. For any two such vector fields \( \mathbf{X}, \mathbf{Y} \), \( [\mathbf{X}, \mathbf{Y}] = 0 \).

Example 6. Let
\[
\begin{bmatrix}
X^1(x) \\
X^2(x)
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = A x
\]

Let
\[
\begin{bmatrix}
Y^1(x) \\
Y^2(x)
\end{bmatrix} = \begin{bmatrix}
0 \\
1
\end{bmatrix} = b
\]

Then the vector fields \( \mathbf{X} = X^1(x) \frac{\partial}{\partial x^1} + X^2(x) \frac{\partial}{\partial x^2} \) and \( \mathbf{Y} = Y^1(x) \frac{\partial}{\partial x^1} + Y^2(x) \frac{\partial}{\partial x^2} \) on \( \mathbb{R}^2 \) with global coordinates \( \phi = (x^1, x^2) \), have the Lie bracket
\[
[X, Y] = [\mathbf{X}, \mathbf{Y}] = \sum_{i=1}^{2} \frac{\partial}{\partial x^1} X^i \frac{\partial}{\partial x^2} Y^i = -A b
\]

where
\[
Z = \begin{bmatrix}
Z^1 \\
Z^2
\end{bmatrix} = -A b
\]

Definition 7. If \( \mathbf{X} \) is a smooth vector field on a smooth manifold \( M \), we say \( \mathbf{X} \) is an integral curve of \( \mathbf{X} \) starting at \( m \), if \( x(0) = m \in M \), and for \( 0 \leq t \leq T \),
\[
X_t(t) = X_{x(t)}(t)
\]
Theorem 8. There is a smooth integral of \( X \) passing through each \( m \).

Proof. We wish to show that integral curves exist for small time and are unique given \( Y(0) = m \). Observe that

\[
Y(t) = X_{\phi(t)}
\]

\[
Y(t)f = X_{\phi(t)}f \quad \forall f \in F(M, \phi(t))
\]

\[
\frac{d}{dt}(f \circ \phi^{-1} \circ \phi \circ Y) = X_{\phi(t)}f \quad \forall f \in F(M, \phi(t))
\]

\[
\begin{align*}
&\text{and } \phi : U \to W \subset \mathbb{R}^k \\
&\quad \quad t \mapsto \phi(p) = (x_1, \ldots, x_k) \\
&\text{is a local chart on } U \ni m
\end{align*}
\]

\[
= \sum_{i=1}^{k} X_i(\phi(t)) \frac{d}{dt} f
\]

\[
= \sum_{i=1}^{k} x_i(\phi(t)) \frac{d}{dt} f
\]

\[
= \sum_{i=1}^{k} x_i' \left( x_i(\phi(t)) \frac{d}{dt} f \right)
\]

(Here \( x_i' \) are components of \( X \) expressed in terms of local coordinates.)

Letting \( f = x_i \), \( i = 1, 2, \ldots, k \), we have equivalence to the ordinary differential equation.
\[
\begin{align*}
\frac{dx}{dt} &= \tilde{x}(x), \quad \text{on } V.
\end{align*}
\]

By smoothness, we have local existence and uniqueness of a solution to the ODE above producing a curve \( \tilde{x} : [0, T] \to V \)
\[
\text{such that } \tilde{x}(0) = x(0) = \varphi(m).
\]

The integral curve \( x = \varphi^{-1} \circ \tilde{x} \).

This completes the proof. \( \square \)

**Example 9**

Let \( SO(n) = \{ A : A^T A = I, \text{det}(A) = 1 \} \subset \text{Mat}(n) \cong \mathbb{R}^{n \times n} \).

Consider a smooth curve \( t \mapsto A(t) \in SO(n) \).

Then
\[
\begin{align*}
\frac{d}{dt} A(t) A(t) + A(t)^T \frac{d}{dt} A(t) &= 0 \\
\Rightarrow \quad (A(t)^T \dot{A}(t))^T + (A(t)^T \dot{A}(t)) &= 0 \\
\Rightarrow \quad \dot{A}(t)^T A(t) &= \Sigma(t) \in \text{SkewMat}(n) \cong \mathbb{R}^{n \times n}
\end{align*}
\]

Support \( \Sigma(t) \) is a given \( \text{smooth} \) skew-symmetric function of \( t \). It is elementary that given \( A(0) \in SO(n) \), there is a unique solution \( A(t) \in SO(n) \) such that, given by the Neumann-Peano-Blow
Series expansions

\[ A(t) = A(0) \left( 1 + \int_0^t \omega_1(s) \, ds + \int_0^t \int_0^s \omega_2(r) \, dr \, ds + \cdots \right) \]

The differential equation \( \dot{A}(t) = A(t) \omega(t) \)
defines a vector field in \( \text{Mat}(n) \) which restricts to a vector field in \( \text{SO}(n) \).

**Remark 10.** It is perhaps) to explicitly write down the tangent space \( T_{SO(n)} \) at \( A \in SO(n) \) and recognize that the right-hand side of the differential equation in Example 9 takes values in \( T_{SO(n)} \). For this one can use a corollary of the pre-image theorem:

Suppose \( f : X \to Y \) is a map of smooth manifolds and \( y \in Y \) is a regular value of the map in the sense that \( (df)_x : T_x X \to T_y Y \) is surjective at every \( x \in f^{-1}(y) \). Then the tangent space to the manifold \( f^{-1}(y) \)
at \( x \), \( T_x f^{-1}(y) = \text{Ker}(df)_x \subset T_x X \).

Let \( f : SO(n) \to \mathbb{H} \to \mathbb{R} \)

\[ A \mapsto AA^T - I \]
Let \( f : \text{Mat}(n) \to \text{Sym Mat}(n) \cong \mathbb{R}^{n(n+1)/2} \)
\( A \mapsto A^T A - I \)

Then \( O(n) = f^{-1}(0) \) — the zero matrix.

\[
(df)_{\mathcal{A}} : \text{Mat}(n) \to \text{Sym Mat}(n)
H \mapsto (df)_{\mathcal{A}} : H = \left. \frac{d}{dt} f(A + th) \right|_{t=0}
= (A + th)(A + th)^T + (A + th)^T(A + th)
\left. \right|_{t=0}
= H^T A + A^T H
\]

\((df)_{\mathcal{A}}\) is surjective for every \( A \in f^{-1}(O) \) since any symmetric matrix \( C \) can be written as
\( C = H^T A + A^T H \)
for some \( H \). For instance, pick \( H = \frac{1}{2} AC \).

Thus \( O \) is a regular value of \( f \). Now
\( f^{-1}(O) = O(n) \) is a smooth manifold by the pre-image theorem.

\[
\text{Ker}(df)_{\mathcal{A}} = \{ H : H^T A + A^T H = 0 \}
= \{ H : (A^T H)^T + A^T H = 0 \}
\]

Thus \( H \in \text{Ker}(df)_{\mathcal{A}} \iff A^T H \) is skew symmetric.

\[
\iff H = \Omega A \quad \text{for some skew symmetric matrix} \; \Omega
\]

Thus we have shown that the right hand side of the differential equation in Example 9 takes values in \( TSO(n) \) when \( A \in SO(n) \).
Remark 11. Any $\Psi \in F(O(n), A)$ can be extended smoothly to $\Psi \in F(Mat(n), A)$, since $O(n) \subset Mat(n)$. For any smooth curve $\gamma: \mathbb{R} \rightarrow O(n)$, $\gamma(0) = A$, such that $\gamma'(0) = A$,

\[
\gamma(t) = A e^{t \Psi(t)} \bigg|_{t=0} = \frac{d}{dt} \gamma(t) \bigg|_{t=0} = (D \gamma)(A)
\]

\[
A \frac{d}{dt} \gamma(t) = A e^{t \Psi(t)} \quad \text{for } \Psi \text{ a skew-symmetric matrix. We have a representation of tangent vectors to } O(n) \text{ at } A \text{ as matrices } A \Psi.
\]

Definition 12. The **flow** of a vector field $X$ on a manifold $M$ is the collection of maps $\Phi^t: M \rightarrow M$ such that $\Phi^0(x) = x$ is the unique integral curve of $X$ with initial condition $x \in M$. Existence and uniqueness theorems from ordinary differential equations ensure that $\Phi^t$ is smooth in $t$ and $x$. It also follows that

\[
\Phi^{t+s} = \Phi^t \circ \Phi^s \quad \text{(flow property)}
\]

and $\Phi^0 = id_M$.

Thus, $\Phi_t^{-1} = (\Phi_t)^{-1}$ is the smooth inverse of the diffeomorphism $\Phi_t$. 
The Kinematic Car

Consider a front-wheel drive car as in the figure.

The base line $AB$ is of fixed length $l$. The point $B$ is located at $(x,y)$ at the instant of observation. Heading $\alpha$ and $\theta$ is the steering angle.

At any instant, the configuration of the car is given by $(x, y, \alpha, \theta) \in \mathbb{R}^2 \times T^2 = \text{configuration space}$.

There are two distinguished vector fields on configuration space:

\[
\text{steer} = \frac{2}{\partial \theta}
\]

\[
\text{drive} = \cos(\phi + \theta) \frac{2}{\partial x} + \sin(\phi + \theta) \frac{2}{\partial y} + \frac{1}{l} \sin(\theta) \frac{2}{\partial y}
\]

Convince yourself that

\[
[\text{steer, drive}] = -\sin(\phi + \theta) \frac{2}{\partial x} + \cos(\phi + \theta) \frac{2}{\partial y} + \frac{1}{l} \cos(\theta) \frac{2}{\partial y}
\]
Let \( \text{slide} = -\sin(\phi) \frac{\partial}{\partial x} + \cos(\phi) \frac{\partial}{\partial y} \)

\[ \text{rotate} = \frac{1}{i} \frac{\partial}{\partial \phi} \]

Verify that at \( \phi = 0 \)

\[ [\text{steer, drive}] = \text{slide} + \text{rotate} \]

We call the bracket \( [\text{steer, drive}] \), "wriggle."

Check

\[ [\text{steer, wriggle}] = -\text{drive} \]
\[ [\text{wriggle, drive}] = \text{slide} \]

Additionally, "slide" commutes (i.e., has vanishing Jacobi-Lie bracket) with steer, drive and wriggle.

Nelson pointed out the "parking algorithm": wriggle, drive, reverse wriggle, reverse drive, wriggle, drive,

A key question is how to we implement the Jacobi-Lie bracket "wriggle" if we have authority only over steer/drive? To understand this, the interpretation in the following example is of use.

**Example 13.** Consider the time-dependent differential equation in \( \mathbb{R}^n \):

\[ \dot{x} = u(t) A x + v(t) B x \]

where the control signals \( u(t) \) and \( v(t) \) are chosen
in such a way as to manipulate the trajectory of $x$.
The vector fields $X = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} (A x_i) \frac{\partial}{\partial x_i}$
and $Y = \sum_{i=1}^{n} y_i \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} (B x_i) \frac{\partial}{\partial x_i}$
are modulated by the control signals. Consider the choice of control signals:

$u(t) = 1, \quad 0 \leq t \leq \varepsilon; \quad u(t) = 0, \quad \varepsilon \leq t \leq 2\varepsilon;$$u(t) = -1, \quad 2\varepsilon \leq t \leq 3\varepsilon; \quad u(t) = 0, \quad 3\varepsilon \leq t \leq 4\varepsilon$

and

$v(t) = 0, \quad 0 \leq t \leq \varepsilon; \quad v(t) = 1, \quad \varepsilon \leq t < 2\varepsilon;$$v(t) = 0, \quad 2\varepsilon \leq t \leq 3\varepsilon; \quad v(t) = -1, \quad 3\varepsilon \leq t \leq 4\varepsilon.$

These are exhibited in the figure:

The motion in state space is captured in the diagram:

$X_0$ is the initial state.
Thus \( x(\epsilon) = e^{-\epsilon A} e^{\epsilon B} e^{\epsilon A} x_0 \). What is the gap \( x(\epsilon) - x_0 \)? By expanding the flows (exponentials) in series, we get

\[
x(\epsilon) - x_0 = e^{\epsilon (B - A^2)} x_0 + O(\epsilon^2).
\]

The leading term \( (0 - A^2) x_0 = L x_0 \) viewed as a vector in \( \mathbb{R}^n \). We have implemented (approximately due to the \( O(\epsilon^2) \) term), the Jacobi-Lie bracket of \( X \) and \( Y \) by resorting to the switching controls \( u(\cdot), v(\cdot) \).

It is a remarkable outcome of Taylor expansions that even if the linear vector fields above are replaced by arbitrary smooth vector fields, the formula

\[
x(\epsilon) - x_0 = e^\epsilon [X, Y] |_{x_0} + O(\epsilon^2)
\]

is true.

Here we have identified the tangent space at \( x_0 \) with \( \mathbb{R}^n \) as the state space.