More examples of manifolds

(A) Sphere $S^{n-1} = \{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 = 1 \}$

The stereographic projection for charts.

$N = (0, 0, 0, \ldots, 1)$ is the 'north pole',

$S = (0, 0, 0, \ldots, -1)$ is the 'south pole'.

$\phi : S^{n-1} \setminus \{N\} \to \mathbb{R}^{n-1}$

\[ (x_1, x_2, \ldots, x_n) \mapsto \frac{1}{1 - x_n} (x_1, x_2, \ldots, x_{n-1}) \]

$\phi : S^{n-1} \setminus \{S\} \to \mathbb{R}^{n-1}$

\[ (x_1, x_2, \ldots, x_n) \mapsto \frac{1}{1 + x_n} (x_1, x_2, \ldots, x_{n-1}) \]

$\phi_N$ and $\phi_S$ are homeomorphisms onto $\mathbb{R}^{n-1}$ and define coordinate charts $(S^{n-1} \setminus \{N\}, \phi_N)$ and $(S^{n-1} \setminus \{S\}, \phi_S)$ respectively. These charts cover $S^{n-1}$ and they overlap.

$\phi \circ \phi^{-1} : \mathbb{R} \to \mathbb{R}^{n-1}$

\[ y \mapsto \frac{y}{\sqrt{1 + y^2}} \]

is a diffeomorphism of punctured $\mathbb{R}^{n-1}$

( Check this formula )
where $\text{Res}(q(c), p(c))$ stands for the resultant determinant

$$
\begin{vmatrix}
1 & b_{n-1} & \cdots & b_0 \\
0 & 1 & b_{n-1} & \cdots & b_0 \\
0 & 0 & 1 & \cdots & b_0 \\
\vdots & & & \ddots & \ddots \\
0 & \cdots & & & 1 \\
q_{n-1} & q_{n-2} & \cdots & q_0 & 0
\end{vmatrix}
$$

Thus, we identify $\text{Rat}(n)$ as the complement in $TR^{2n}$ (viewed as the space of coefficients $q_i, p_j$), of the resultant locus

$$
[\text{Res}(q(c), p(c))] = \{ \frac{q(c)}{p(c)} : \text{Res}(q(c), p(c)) = 0 \}
$$

But the resultant locus is a closed set in $TR^{2n}$ (because the determinant of a matrix is a polynomial in the elements of the matrix). Thus,

$$
\text{Rat}(n) = TR^{2n} - [\text{Res}(q(c), p(c))]
$$

is an open submanifold of $TR^{2n}$.

(see example 3 on page 18 of Lecture 1.)
The case $n = 1$:

$$\text{Rat (1)} = \left\{ \frac{v_0}{s + t_0} : (v_0, t_0) \in \mathbb{R}^2, v_0 \neq 0 \right\}$$

$\mathbb{R}^2$

$\frac{q_0}{s}$

$q_0 = 0$ axis = resultant locus

splits the space $\text{Rat (1)}$ into two disconnected pieces, each diffeomorphic to $\mathbb{R}^2$. For $n = 2$ and beyond, $\text{Rat (n)}$ is quite an interesting space. For instance,

$$\text{Rat (2)} \cong \mathbb{R}^4 \cup S^1 \times \mathbb{R}^3 \cup \mathbb{R}^4$$

is diffeomorphic to the disjoint union of two copies of $\mathbb{R}^4$ and a copy of the product manifold $S^1 \times \mathbb{R}^3$.

Here $S^1$ denotes the circle.

Note, cartesian products of manifolds are manifolds.

There are very interesting dynamical systems on $\text{Rat (n)}$ — we will see later.
Remark 1. A smooth bijective map of manifolds need not be a diffeomorphism. 

Example \( f : \mathbb{R}^1 \to \mathbb{R}^1 \)
\[ x \mapsto x^3 \]
has a continuous inverse
\( g : \mathbb{R}^1 \to \mathbb{R}^1 \)
\[ y \mapsto y^{1/3} \]
but \[ \frac{dg}{dy} = \frac{1}{3} y^{-2/3} \] is not defined at \( y = 0 \).

Remark 2. The union of two coordinate axes in \( \mathbb{R}^2 \) is not a manifold. 

\[ \text{WHY?} \]
Definition 3 Let $M^k$ be a smooth manifold in the sense of the abstract definition (Definition 23, Lecture Notes 1, page 13–16). Then $f : M \to \mathbb{R}$ is a smooth function, if for each local parametrization $(\omega_d, \eta_d)$,

$$f \circ \eta : \omega_d \subset \mathbb{R}^k \to \mathbb{R}$$

is smooth.

If $M^k$ is given as a subset of $\mathbb{R}^N$, then the definition of smoothness above agrees with the Definition 20, of Lecture Note 1, page 12.

We are interested in defining tangent spaces to manifolds. There are several equivalent ways of doing this. All need proper notions of derivatives.
Suppose $X$ is a vector space and $Y$ is a normed linear space with norm $\| \cdot \|_Y$. Then, for $x \in U \subseteq X$, let $T: U \rightarrow Y$ be a given map. For $h \in X$, we define a Gateaux differential of $T$ at $x$ with increment $h$ to be a map (if it exists) $S_T : X \rightarrow Y$

$$h \mapsto S_T(x; h) = \lim_{\alpha \to 0} \frac{T(x + \alpha h) - T(x)}{\alpha}$$

where the limit is taken in $\| \cdot \|_Y$. This is a generalization of the concept of directional derivative.

Suppose $X$ and $Y$ are both normed linear spaces. Let $T: U \subseteq X \rightarrow Y$ be a given map and $x \in U$. We define the Frechet differential of $T$ at $x$ with increment $h$, if it exists, to be the linear map $D_T: X \rightarrow Y$

$$h \mapsto D_T(x; h)$$

satisfying

$$\lim_{\| h \|_X \to 0} \frac{\| T(x + h) - T(x) - D_T(x; h) h \|_Y}{\| h \|_X} = 0.$$ 

Remark 4. If $T$ is Frechet differentiable, then it is Gateaux differentiable, and the two notions agree.

Exercise 5. (a) If the Frechet derivative $D_T(x; \cdot)$ exists, it is unique.
Use the short-hand, \( DT(x) \) for \( DT(x; \cdot) \).

(b) Let \( f: \mathbb{R}^n \to \mathbb{R}^m \) and \( g: \mathbb{R}^m \to \mathbb{R}^p \) be differentiable.

Then \( D (g \circ f)(x) = D g(f(x)) \cdot D f(x) \) (chain rule).

We also use \( dT_x \) for \( DT(x) \).

To understand what the differential looks like in coordinates, let

\[ f: \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^m \]

be a smooth map.

Then \( df_x: \mathbb{R}^n \to \mathbb{R}^m \)

\[ h \mapsto df_x(h) = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t} = \frac{df(x+th)}{dt} \bigg|_{t=0} \]

Linearity of \( df_x \) implies there is a matrix representation.

With coordinates \((x_1, \ldots, x_n)\) on \( \mathcal{U} = \mathbb{R}^n \) and \((y_1, \ldots, y_m)\) on \( \mathbb{R}^m \), we write

\[ f = (f_1, \ldots, f_m), \quad f_i = f_i(x_1, \ldots, x_n) \]

Then \( df_x(h) = A \cdot h \) where

\[ A \text{ is the } m \times n \text{ matrix } A = \left[ \frac{\partial f_i}{\partial x_j} \right] \]

where all the derivatives are evaluated at the point \( x = (x_1, \ldots, x_n) \).

Exercise 6(c) Suppose we change coordinate in \( \mathbb{R}^n \)

to \((\tilde{x}_1, \ldots, \tilde{x}_n)\), \( \tilde{x}_i = \phi_i(x_1, \ldots, x_n) \) \( i=1, 2, \ldots, n \),

What is the matrix \( A \) in the new coordinates?
q.

Since the composition of linear maps simply corresponds to matrix multiplication, we write the chain rule as a commutative diagram.

\[ \text{Given} \]
\[ f: U \subset \mathbb{R}^n \rightarrow V \subset \mathbb{R} \]
\[ g: V \rightarrow \mathbb{R}^l \]
\[ g \circ f \]

Both smooth maps,

(Continued) on page 10

Exercise 6(b) Let \( \iota: U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) be the inclusion map. Then show that \( \iota_x: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is identity map.
We have (the chain rule) the diagram of vector space homomorphisms.

\[
\begin{array}{ccc}
\mathbb{R}^k & \xrightarrow{d\varphi(x)} & \mathbb{R}^l \\
\mathbb{R}^n & \xrightarrow{df_x} & \mathbb{R}^m
\end{array}
\]

With this set-up we can now define a tangent space. Let \( X \) be a manifold. Let \( x \in X \) and let \( \eta : W \rightarrow \mathbb{R}^k \rightarrow X \) be a local parametrization at \( x \), i.e. \( x \in U = \eta(W) \). Assume, without loss of generality, that \( 0 \in W \) and \( \eta(0) = x \). The linear approximation to \( \eta \) at \( 0 \) is the map \( u \mapsto \frac{d\eta(0)}{d\eta(W)} u + d\eta(W) \).

[This makes sense when we think of \( x \in \mathbb{R}^N \).]

**Definition 7** The tangent space to \( X \) at \( x \) is the image of the linear map \( d\eta : \mathbb{R}^k \rightarrow \mathbb{R}^N \).

**Question 8** Is the tangent space well-defined, i.e. is it independent of the choice \( (W, \eta) \) at \( x \)? To answer this question it is useful to know,

**Proposition 9** Suppose \( f : \mathbb{R}^m \rightarrow \mathbb{R}^m \) is a diffeomorphism. Then

\[
d(f^{-1}) = [d_f^{-1}(x) f]^{-1}.
\]
Proof: Use the fact that
\[ (f \circ f^{-1})(x) = x \]
\[ \forall x \in \mathbb{R}^m. \text{ Then use chain rule.} \]

**Lemma 10** The tangent space \( T_x \mathcal{X} \) at \( x \in \mathcal{X} \)

is independent of the choice of local parametrization.

**Proof**

Let \( U = U_1 \cap U_2 \)
Let \( S_1 = \eta_1^{-1}(U) \); \( S_2 = \eta_2^{-1}(U) \)
\[ \subset W_1 \subset W_2 \]

Let \( \psi = \eta_1^{-1} \circ \eta_2 : S_2 \to S_1 \).
\( \psi \) is a diffeomorphism.
Therefore
\[ \eta_0 \cdot \eta^{-1} = \eta \]
\[ \frac{dy}{dy} \cdot \frac{dy}{dy} = \frac{dy}{dy} \]
(by chain rule)

with derivatives maps evaluated at appropriate points.

Thus,
\[ \text{im} \left( \frac{dy}{dy} \right) \subseteq \text{im} \left( \frac{dy}{dy} \right) \]

By a similar argument
\[ \text{im} \left( \frac{dy}{dy} \right) \subseteq \text{im} \left( \frac{dy}{dy} \right) \]

Thus,
\[ T_X = \text{im} \left( \frac{dy}{dy} \right) \subseteq \text{im} \left( \frac{dy}{dy} \right) \]

Lemma 11. \[ \dim \left( \text{T}_X \right) = \dim \left( X \right) = k. \]

Proof. Let \((W, \eta)\) be a local parameterization at \(x \in X\). \[ \eta : W \subseteq \mathbb{R}^k \rightarrow X \subset \mathbb{R}^N \]
\[ \text{im} \left( \eta \right) = U \text{ is open in the topology of } X. \]
Let \( \bar{U} \subseteq \mathbb{R}^N \) be such that \( U = \bar{U} \cap X \) and \( \phi = \eta^{-1} : \bar{U} \rightarrow \mathbb{R}^k \) has a smooth extension \( \varphi : \bar{U} \subseteq \mathbb{R}^N \rightarrow W \subset \mathbb{R}^k \)
Then \( \phi \circ \eta : W \rightarrow W \) is the identity map.

Thus \( d\eta \circ dy = \text{identity} \).

It follows that \( dy \) is an isomorphism into. Therefore \( \dim(\text{im } dy) = \dim(\mathbb{R}^k) = k \). \( \square \)

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An alternative approach to the definition of tangent space at \( x \in M \) involves the study of smooth curves in \( M \) passing through \( x \). We would like to do this without the explicit assumption \( M \subseteq \mathbb{R}^n \) for some \( N \).

**Definition 12** If \( M_i \) and \( M_2 \) are smooth manifolds, a map \( f : M_i \rightarrow M_2 \) is said to be smooth if for every two coordinate systems \((U_1, \phi_1)\) on \( M_1 \) and \((U_2, \phi_2)\) on \( M_2 \), the map

\[
\phi_2 \circ f \circ \phi_1^{-1} : \phi_1(U_1) \subseteq \mathbb{R}^r \rightarrow \mathbb{R}^s
\]

is smooth.

For \( f \) to be smooth, it is sufficient that...