Adjoint equations

For the homogeneous system $x'(t) = A(t)x(t)$, we have the associated adjoint system

$$\dot{p}(t) = -A'(t)p(t).$$

From

$$\frac{d}{dt}(p'(t)x(t)) = p'(t)x(t) + p(t)x'(t)$$

$$= (-A'(t)p(t))'x(t) + p'(t)(A(t)x(t))$$

$$= 0$$

it follows that

$$p'(t)x(t) = p'(t_0)x(t_0) + t.$$  

Moreover, writing $p(t) = \Phi(t,t_0)p(t_0)$ we get

$$\dot{p}(t_0)\Phi(t,t_0)\Phi(t_0,x_{t_0}) = p'(t_0)x(t_0) + t$$

Since this is true for arbitrary $x(t_0)$, $p(t_0)$ it follows that

$$\Phi(t,t_0)\Phi(t,t_0) = I + t$$

$$\Rightarrow$$

$$\Phi(t,t_0) = \Phi(t_0,t) + t.$$  

From this we also have a corollary:

$$\frac{d}{dt}\Phi(t_0,t) = \frac{d}{dt}(\Phi(t_0,t))$$

$$= (-A'\Phi)(t,t_0)$$

$$= -\Phi'((t_0,t)A(t))$$

$$= -\Phi((t_0,t)A(t))$$

$$- 1$$
Consider the linear time-varying system (canonical equation)

\[
\begin{pmatrix}
\dot{x}(t) \\
\dot{p}(t)
\end{pmatrix} =
\begin{pmatrix}
A(t) & -B(t)B'(t) \\
-L(t) & -A'(t)
\end{pmatrix}
\begin{pmatrix}
x(t) \\
p(t)
\end{pmatrix}
\] 

\[\tag{C}\]

evolving on \(\mathbb{R}^{2n}\). (Assume \(L(t) = L'(t)\)). Let \(H(t, x, p)\) denote the function \(H: \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}\)

\[
(t, x, p) \mapsto H(t, x, p)
\]

\[
= \frac{1}{2} x' L(t) x + p' A(t) x
\]

\[- \frac{1}{2} p' B(t) B'(t) p
\]

Define the gradient of \(H\), \(\nabla H = \left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial p}\right)\).

\[
\nabla H = 
\begin{pmatrix}
L(t) & A'(t) \\
A(t) & -B(t)B'(t)
\end{pmatrix}
\begin{pmatrix}
x \\
p
\end{pmatrix}
\]

We can then rewrite the given system \((C)\) in the form

\[
\begin{pmatrix}
\dot{x} \\
\dot{p}
\end{pmatrix} = J \nabla H
\]

where \(J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\) is a skew-symmetric invertible matrix. Any trajectory of \((C)\) the derivative of \(H\) w.r.t. time can be computed using chain rule:

\[
\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_{i=1}^{n} \frac{\partial H}{\partial x_i} \dot{x}_i + \frac{\partial H}{\partial p} \dot{p}
\]

\[
= \frac{\partial H}{\partial t} + (\nabla H)'(\dot{x}, \dot{p}) = \frac{\partial H}{\partial t} + (\nabla H)' J \nabla H
\]

\[-2-\]
Since $J$ is skew, the second term on the right is identically zero. If further, the parameters $A, B, L$ are time-invariant, then \( \frac{d\Phi}{dt} = 0 \) and \( \frac{d\Phi^T}{dt} = 0 \)

\[ \Rightarrow H = \text{constant}. \]

What are canonical equations (C) good for?

For one thing, solving Riccati equations.

**Lemma 1**

Let \( \Phi(t, t_0) \) denote the \( 2n\times2n \) transition matrix

for (C). Partition into blocks: \( \Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \)

of size \( n \times n \). Then:

(a) If \( \Phi(t_2, t_1) = \left( \Phi_{22}(t_1, t_2) - Q \Phi_{12}(t_1, t_2) \right) - 1 \\ \left( Q \Phi_{11}(t_1, t_2) - \Phi_{12}(t_1, t_2) \right) \)

or equivalently, if

(b) \( \Phi(t_2, t_1) = \left( \Phi_{21}(t, t_2) + \Phi_{22}(t, t_2) Q \right) \left( \Phi_{11}(t, t_1) + \Phi_{12}(t, t_1) Q \right) \)

then \( \Phi(t_2, t_1, Q, t_1) = Q \) and, \( \Phi(t_2, t_1) \) satisfies the Riccati equation,

\[ \Phi(t_2, t_1) = -A^T \Phi(t_2, t_1) - \Phi(t_2, t_1) A + \Phi(t_2, t_1) B B^T \Phi(t_2, t_1) - L \]

assuming that the indicated inverses exist.

**Proof:** Let, \( [X(t), -P(t)] = [Q, -A] \left[ \begin{pmatrix} \Phi_{11}(t_1, t_2) & \Phi_{12}(t_1, t_2) \\ \Phi_{21}(t_1, t_2) & \Phi_{22}(t_1, t_2) \end{pmatrix} \right] \)

Clearly, in part (a) above, the r.h.s. = \( P^{-1}(t_2) X(t_2) \).
To verify that $T(t, s, t_1) = P^{-1}(t) X(t)$ satisfies the Riccati equation, differentiate:

$$\dot{P}^{-1} X = -P^{-1} \dot{P} P^{-1} X + P^{-1} \dot{X}$$

From the definition of $[X, P]$ we see

$$[X, -P] = [\bar{Q}, -1] \frac{d}{dt} \Phi(t_1, t)$$

$$= [\bar{Q}, -1] \frac{d}{dt} \Phi(t, t_1)$$

$$= [\bar{Q}, -1] \left(-\frac{\Phi^{-1}}{\Phi(t, t_1)} \frac{d}{dt} \Phi(t, t_1) \Phi(t, t_1) \right)$$

$$= -[\bar{Q}, -1] \Phi(t_1, t) \left( \begin{array}{cc} \bar{A}(t) - \bar{B}(t) \bar{B}(t) \\ -L(t) & -A(t) \end{array} \right)$$

$$= -[X, -P] \left( \begin{array}{cc} A & -BB' \\ -L & -A' \end{array} \right)$$

$$\Rightarrow \dot{X} = -XA - PL$$

$$\dot{P} = -XBB' + PA'$$

$$\Rightarrow \frac{d}{dt} (P^{-1} X) = -P^{-1} (-XBB' + PA') P^{-1} X + P^{-1} (XA - PL)$$

$$= P^{-1} XBB' P^{-1} X - A' P^{-1} X - P^{-1} X A - L$$

which is what we set out to prove.

Also $P^{-1}(t) X(t) \bigg|_{t=t_1} = \bar{Q}$ since $\Phi(t_1, t_1) = (\begin{array}{cc} I & 0 \\ 0 & I \end{array})$.

Proof of part (b)

Define $$(\bar{X}(t), \bar{P}(t)) = \left( \begin{array}{c} \bar{X}_{11}(t, t_1) \\ \bar{X}_{12}(t, t_1) \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right).$$

Clearly, in part (b) above $\text{r.h.s} = \bar{P}(t) \bar{X}(t) - 1$. Rest of the steps similar to the steps in part (a) proof.
Remark. Matrices of the form
\[ P = \begin{pmatrix} A & Q \\ R & -A' \end{pmatrix} \]
where each of the blocks is \( n \times n \) and \( A = A', \ R = R' \)
are called infinitesimally symplectic or Hamiltonian matrices.
They satisfy the identity.
\[ P'J + JP = 0 \]

Necessary Conditions for Optimality

Theorem. For the system
\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) \]
\[ x(t_0) = x_0, \] let \( u(t) \) be one of the following controls

(a) \[ u_0(t) = -B(t)F_A^{-1}(t_0, t_1) \xi \] where \( \xi \) satisfies
\[ W(t_0, t_1) \xi = x_0 - F_A^{-1}(t_0, t_1) x_1. \]

(b) \[ u_1(t) = -B(t)T(t, t_1) x(t) \]
\[ T(t) = -A'(t)T(t, t_1) - T(t, t_1)A(t) + L(t) \]
\[ + T(t, t_1)B(t)B(t)T(t, t_1) \]
\[ T(t_1, t_1) = Q \]
and \( L(t) = L(t) + t \)

(c) \[ u_2(t) = -B(t)T(t, K_1, t_1) x(t) + v(t) \]
\[ T(t) = -A'T(t) - T(t)A(t) + T(t)BB'T(t) \]
\[ T(t_1, K_1, t_1) = K_1 \]
and \( v \) such that \( \int_0^t v(s) v(s) ds \) for
\[ \dot{x}(t) = (A(t) - B(t)B(t)T(t, K_1, t_1))x(t) \]
\[ x(t_0) = x_0; \ x(t_1) = x_1. \]
Then, there exists a vector function \( \phi(t) \) (the co-state) such that

\[
\begin{pmatrix}
    \dot{x} \\
    \dot{p}
\end{pmatrix} =
\begin{pmatrix}
    A & -BB' \\
    -L & -A'
\end{pmatrix}
\begin{pmatrix}
    x \\
    p
\end{pmatrix},
\]

\( x(t_0) = x_0 \)

and

\( u(t) = -B(t)p(t) \).

**Proof**

(a) Let \( p(t) = \Phi^{-1}_A(t_0, t)\phi' \).

Then \( \dot{p} = -A(t)p \)

with \( p_0 = \phi' \) (from page 1 of this lecture).

Substituting \( u_0 \) in the state equation, we get

\[
\dot{x} = Ax - BB'p
\]

Picking \( L = 0 \) ensures that \( \dot{p} = -A'p = -Lx - A'p \).

This completes the proof of part (a).

(b) Let \( p(t) = \Pi(t, \xi, t_1)x(t) \).

Then substituting \( u_1 \) in the state equation, we get

\[
\dot{x} = Ax - BB'p
\]

We need to show

\[
\dot{p} = -Lx - A'p
\]

Differentiate \( \Pi(t, \xi, t_1)x(t) \) to get,

\[
\dot{p} = \Pi x + \Pi \dot{x} = (\Pi A' - \Pi A - L)x + \Pi BB' \dot{x} + \Pi (Ax + BB'x)
\]
\[ = -A'p - Lx \]

The boundary condition on \( T \) turns into
\[ p(t_1) = Qx(t_1). \]

(c) Left as an exercise

Remark: We postpone discussion of the infinite horizon optimal control problem and associated algebraic Riccati equations.

Using the Canonical Equation

From proof of part (a) of the Theorem, it is clear that solving (c) for \((x_0, p_0)\), initial conditions, sweeps out a 'bundle' of state/costate trajectories as \( p_0 \) is varied. Only \( p_0 \), s.t.
\[ W(c_0, t_1) = x_0 - \int^T V(t_0, t) dt \]
will produce trajectories/trajectories satisfying end-point conditions. End-point error associated to a given \( p_0 \) can be used to correct \( p_0 \). Similar remarks apply to cases (b) & (c).

Analogue of (c) play a central role in general optimal control problems (not necessarily linear systems with quadratic cost functionals): We will encounter these later.