An example on Fréchet derivatives.

Let $f: \mathbb{R}^n \to \mathbb{R}$ have continuous first partial derivatives $\frac{\partial f}{\partial x_i}$. Then

$$f'(x; h) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i,$$

is also the Fréchet derivative $Df(x; h)$. □

Clearly, the row vector $(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ defines a bounded linear operator on $\mathbb{R}^n$. We need to show the limit property

$$\lim_{h \to 0} \frac{1}{\|h\|} \|f(x+h) - f(x) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i\| = 0.$$

Equivalently, we need to show that, given any $\epsilon > 0$, there exists $\delta > 0$ such that whenever $\|h\| < \delta$,

$$\frac{1}{\|h\|} \|f(x+h) - f(x) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i\| < \epsilon.$$

Equivalently,

$$\|f(x+h) - f(x) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i\| < \epsilon \|h\|.$$

To see this, first observe that, from the hypothesis of continuity of first partial derivatives, given $\epsilon > 0$, there exists $\delta > 0$ s.t.

$$y \in B_{\frac{\delta}{\epsilon}}(x) \iff \|y-x\| < \frac{\delta}{\epsilon}.$$ 

Thus,

$$\|f(x+h) - f(x) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i\| < \epsilon \|h\| = \epsilon \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i} (y) - \frac{\partial f}{\partial x_i} (x) \right| \|h_i\| < \frac{\epsilon}{\epsilon} \sum_{i=1}^{n} \|h_i\| < \frac{\epsilon}{\epsilon} \sum_{i=1}^{n} \|h_i\| = \epsilon \|h\|.$$
Let $\delta = \min_{i=1,2,\ldots,n} \delta_i$. Then,

$$y \in B(x) \Rightarrow \| \frac{\partial f}{\partial x_i}(y) - \frac{\partial f}{\partial x_i}(x) \| < \frac{\delta}{n}$$

for $i = 1, 2, \ldots, n$.

Let $e_1, e_2, \ldots, e_n$ be the standard basis column vector in $\mathbb{R}^n$ with $e_i$ having only the $i$th element nonzero = 1.

Then

$$k = \sum_{i=1}^{n} h_i e_i$$

define $g_0 = 0$ and

$$g_k = \sum_{i=1}^{k} h_i e_k$$

$k = 1, 2, \ldots, n$. Thus $g_n = h$.

Suppose $\| h \| < \delta$

**Note:** On the board, on Thursday Feb 26, I kept on writing $e_k$'s as row vectors. I should have written column vectors.

Observe that for $\mathbb{R}^n$ with their norm $\| x \| = \sum_{i=1}^{n} |x_i|$,

$$\| g_k \| = \sum_{i=1}^{k} |h_i| \leq \sum_{i=1}^{n} |h_i| = \| h \| + h$$

Then

$$f(x + g_k) = f(x) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) h_i$$

$$= \frac{1}{h_k} \sum_{k=1}^{n} (f(x + g_k) - f(x + g_{k-1}) = \frac{\partial f}{\partial x_k}(x) h_k)$$

$$\leq \sum_{k=1}^{n} \left| f(x + g_k) - f(x + g_{k-1}) - \frac{\partial f}{\partial x_k}(x) h_k \right|$$

(b by triangle inequality)

$x + g_k = x + g_{k-1} + h_k e_k$. Below, assume $h_k > 0$. In the mean.

Consider the function $F: [0, h_k] \rightarrow \mathbb{R}$

$$F(z) = f(x + g_{k-1} + z e_k)$$

Then $F(h_k) = f(x + g_{k-1} + h_k e_k) = f(x + g_k)$
Recall the mean value theorem \( MVT \).

Given \( F : [a, b] \to \mathbb{R} \) if continuous on \([a, b]\)
and differentiable on \((a, b)\), there is a \( \xi \) such that \( a < \xi < b \)

such that \( F(b) - F(a) = F'(\xi) \cdot (b-a) \).

Apply \( MVT \) to \( F : [0, h_x] \to \mathbb{R} \) defined on page 2.

Thus there exists \( \xi \), \( 0 < x < h_x \) such that

\[
\begin{align*}
  f(x + g_k) - f(x + g_{k-1}) & = f(x + g_{k-1} + h_k e_k) - f(x + g_{k-1}) \\
  & = \frac{\partial f}{\partial x_k} (x + g_{k-1} + \alpha e_k) \cdot h_k
\end{align*}
\]

Thus \( f(x + g_k) - f(x + g_{k-1}) = \frac{\partial f}{\partial x_k} (x) h_k \)

\[
\begin{align*}
  & = \left( \frac{\partial f}{\partial x_k} (x + g_{k-1} + \alpha e_k) - \frac{\partial f}{\partial x_k} (x) \right) h_k \\
  \text{Clearly } \|x + g_{k-1} + \alpha e_k\| - x & = \sum_{i=1}^{k-1} h_{i-1} + \alpha \\
  & < \frac{k-1}{2} h_{x-1} \\
  & < \|h_k\| < 5
\end{align*}
\]

Thus \( x + g_{k-1} + \alpha e_k \in B(x) \)

\[
\implies \left| \frac{\partial f}{\partial x_k} (x + g_{k-1} + \alpha e_k) - \frac{\partial f}{\partial x_k} (x) \right| < \frac{\|e_k\|}{h_k}
\]

(Attend to the above to see that)

Thus \( \left| f(x + g_k) - f(x + g_{k-1}) - \frac{\partial f}{\partial x_k} (x) h_k \right| < \frac{\|e_k\|}{h_k} \left| h_k \right| \leq \frac{\|e_k\|}{h_k} \|h_k\| \)
The assumption $h_k > 0$ can be replaced by $h_k < 0$ and interchanging $b$ and $a$ in MVT gives the same answer. For $h_k = 0$ it is trivially true. So we find that

$$1 f(x + h) - f(x) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} h_k$$

$$\leq \sum_{k=1}^{n} \left| \frac{\partial f}{\partial x_k} \right| h_k$$

$$= \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} h_k$$

whenever $\|h\| < 5$

This completes the proof.

**EXERCISE** (due back in class March 9, Thursday)

Let $g : \mathbb{R}^2 \to \mathbb{R}$

$$(x, t) \mapsto g(x, t)$$

have a continuous in $x, t$ first partial $g_x(x, t) = \frac{\partial g}{\partial x}(x, t)$.

Let $f : C([0, 1]) \to \mathbb{R}$

$$x \mapsto \int_0^1 g(x(t), t) \, dt$$

Then the Gateaux differential $Df(x; h) = \int_0^1 \frac{\partial g}{\partial x}(x(t), t) \, h(t) \, dt$ is also the Fréchet differential $Df(x; h)$. The norm
on $C[0,1]$ is the infinity norm

$$\|x\|_\infty = \max_{0 \leq t \leq 1} |x(t)|$$