that \( \lim_{a \to 0} \frac{T(x+ah) - T(x)}{a} = DT(x; h) \)

i.e.
\[
ST(x; h) = DT(x; h)
\]

Proposition (continuity from differentiability)

If \( T: U \subseteq X \to Y \) is Frechet differentiable at \( x \), then \( T \) is continuous at \( x \), \( \text{ (here } x \in U) \).

Proof

Given \( \varepsilon > 0 \), there is a ball centered at \( x \) of radius \( \varepsilon \): \( B_{\varepsilon}(x) = \{ \bar{x} \in U : \|\bar{x} - x\| < \varepsilon \} \subseteq U \)

provided \( \varepsilon \) is sufficiently small \( \text{ (since } U \text{ is open).} \)

For \( x + h \in B_{\varepsilon}(x) \),

\[
\|T(x+h) - T(x) - DT(x; h)\| \leq \varepsilon \|h\|
\]

Thus \( \|DT(x+h) - T(x)\| \leq \varepsilon \|h\| + \|DT(x; h)\|\)

\[
\leq \varepsilon \|h\| + \|DT(x; \cdot)\| \|h\|
\]

\[
\leq (\varepsilon + \|DT(x; \cdot)\|) \|h\|
\]

So pick \( \delta = \varepsilon / M \) to get continuity of \( T \).

Example

Let \( f: \mathbb{R}^n \to \mathbb{R} \) have continuous first partial derivatives at \( x_0 \in \mathbb{R}^n \). Then the differential
\[
\delta f(x_0; h) = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \right)_{x=x_0} h_i \text{ is the Frechet differential.}
\]