1. Consider the optimal control problem:

\[
\begin{align*}
\min \quad & \int_0^1 \left( u_1^2(t) + u_2^2(t) \right) dt \\
\text{subject to} \quad & \\
\dot{x}_1(t) = u_1(t) \\
\dot{x}_2(t) = u_2(t) \\
\dot{x}_3(t) = x_1^2(t) u_2(t) - x_2^2(t) u_1(t) \\
x_1(0) = x_1(1) = 0 \\
x_2(0) = x_2(1) = 0 \\
x_3(0) = 0 \quad x_3(1) = 1
\end{align*}
\]

Show that this problem reduces to solving the anharmonic oscillator equation:

\[
\ddot{w} + c_1 \dot{w} + c_2 w^3 = 0
\]

for suitable constants \( c_1 \) and \( c_2 \).

2. Consider the functional

\[
g(x) = \int_0^1 \left( x_1^2(t) \dot{x}_2(t) - x_2^2(t) \dot{x}_1(t) \right) dt
\]

defined on \( X = \{ x: [0, 1] \to \mathbb{R}^2 \mid x \text{ continuous and differentiable and} \}

with suitable norm.
Define $\Omega = \{ x(\cdot) \in X \mid g(x) = 0 \sum_{i=1}^{2} x_i(0) = 0; x_i(1) = 1 \}$

Under what conditions is a given $x \in \Omega$ a regular point of $\Omega$?

3. Consider the problem:

Minimize $\int_0^1 \left( \frac{1}{2} x(t)^2 + \frac{1}{2} x(t)^2 - t x(t) \right) dt$

$x(0) = 0$;
$x(1) = 1$

Does this problem have a unique extremal? If so, is it a minimum of minimum length?

4. Show that a curve minimizing the length joining two points on a sphere

$S^2 = \{ (x, y, z) : x^2 + y^2 + z^2 = 1 \}$

is on a great circle on the sphere.

5. Find (the) point in $\mathbb{R}^2$ that is closest to the origin and also lies on the ellipse (i.e. $A, B > 0$)

$$\frac{(x-a)^2}{A^2} + \frac{(y-b)^2}{B^2} = 1.$$ 

Clearly state any second order conditions.
you use. You may refer to Professor Andoe
Tito's lecture notes for guidance.

6. Consider the problem of minimizing

\[ J[x] = \int_{t_1}^{t_2} L(t, x(t), x'(t)) \, dt \]

with \( t_1 \) and \( t_2 \) given, and

\[ x(t_1) = x_1, \quad \text{but} \quad x(t_2) \quad \text{unspecified}. \]

Derive first order necessary conditions for this problem.

7. It is desired to show that

\[ \int_{t_1}^{t_2} \left( x(t)^2 \right) \, dt + \int_{t_1}^{t_2} \left( x'(t)^2 \right) \, dt \]

for \( x(0) = 0 \)

Solve this problem using

(a) optimal control theory

(b) calculus of variations