Fixed Points

Theorem: Let X be a Banach space and let S ⊆ X be a closed subset. Let T : S → S be a contraction map, i.e. there is a number 0 ≤ p < 1 such that

\[ \| T(x) - T(y) \| \leq p \| x - y \| \quad \forall x, y \in S. \]

Then there is a unique \( x^* \in S \) such that \( x^* = T(x^*) \) (i.e. \( x^* \) is a fixed point of \( T \)). Further this fixed point can be obtained as the limit of a sequence of successive approximations (Banach iteration).

Proof: Let \( x_1 \in S \). Define the sequence \( \{x_n : n \geq 1\} \subseteq S \) by \( x_{n+1} = T(x_n) \).

Thus,

\[ \| x_{k+1} - x_k \| = \| T(x_k) - T(x_{k-1}) \| \]
\[ \leq p \| x_k - x_{k-1} \| \]
\[ \leq p^2 \| x_{k-1} - x_{k-2} \| \] (repeating previous step)
\[ \vdots \]
\[ \leq p^{k-1} \| x_2 - x_1 \|. \]

Hence \( \| x_{k+1} - x_k \| = \| \sum_{i=1}^{k} (x_{k+i} - x_{k+i-1}) \| \)
\[ \leq \sum_{i=1}^{k} \| x_{k+i} - x_{k+i-1} \|. \]

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\[ \sum_{i=1}^{n} \| x_{i} - x_{i-1} \|^p \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \]

\( \therefore \) \[ \{ x_n \} \text{ is a Cauchy sequence. Since } X \text{ is a Banach space, there exists } x^* \in X \text{ such that} \]

\[ x^* = \lim_{n \to \infty} x_n. \] But \( \{ x_n \} \subset S \), \( \therefore \) \( x^* \in S. \)

\[ \| x^* - T(x^*) \|^p = \| x^* - x_n + x_n - T(x^*) \|^p \]

\[ \leq \| x^* - x_n \|^p + \| x_n - T(x^*) \|^p \]

\[ = \| x^* - x_n \|^p + \| T(x_{n-1}) - T(x^*) \|^p \]

\[ \leq \| x^* - x_n \|^p + \rho \| x_{n-1} - x^* \|^p \]

\[ \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \]

Here \( \| x^* - T(x^*) \|^p = 0 \iff x^* = T(x^*). \)

Suppose there is a \( y^* \in S \) also a fixed point of \( T \).

\[ \| x^* - y^* \|^p = \| T(x^*) - T(y^*) \|^p \]

\[ \leq \rho \| x^* - y^* \|^p \]

\[ \rho < 1 \quad \therefore \quad \| x^* - y^* \|^p = 0 \iff x^* = y^*. \]

We have shown uniqueness.
If instead of a specific map $T$, we have a parametrized family of maps, then we can show continuity of the fixed point w.r.t. $\theta$. For such a family $T: \Theta \times S \rightarrow S$, define

$$T_\theta: S \rightarrow S \quad \text{by} \quad T_\theta(x) \equiv T(\theta, x) \quad x \in S$$

and

$$T^x: \Theta \rightarrow S \quad \text{by} \quad T^x(\theta) \equiv T(\theta, x) \quad \theta \in \Theta.$$ 

These maps $T_\theta$ and $T^x$ are called partial maps associated to the family $T$.

**Theorem 2:** Let $\Theta$ be a metric space with metric $d$. Let $X$ be a Banach space and let $S \subseteq X$ be a closed subset such that the family $T: \Theta \times S \rightarrow S$ has the following properties

(i) Each partial map $T_\theta: S \rightarrow S$, $\theta \in \Theta$, is a contraction with contraction coefficient $p < 1$

(ii) Each partial map $T^x: \Theta \rightarrow S$, $x \in S$, is continuous, i.e. given $\varepsilon > 0$, there exists $\delta_x > 0$ such that,

$$d(\theta, \theta') < \delta_x \Rightarrow \|T^x(\theta) - T^x(\theta')\| < \varepsilon.$$

Then the map $F: \Theta \rightarrow S$, $F(\theta) \equiv x^*_\theta = \text{unique fixed point of } T^x_\theta$, is continuous.

**Proof:**

$$\|x^*_\theta - x^*_\theta'\| = \|T_\theta(x^*_\theta) - T_\theta(x^*_\theta')\|$$

$$\leq \|T(x^*_\theta) - T(x^*_\theta')\| + \|T_\theta(x^*_\theta) - T_\theta(x^*_\theta')\|$$

$$\leq p \|x^*_\theta - x^*_\theta'\| + \|T_\theta(x^*_\theta) - T_\theta(x^*_\theta')\|$$

$$\leq p \|x^*_\theta - x^*_\theta'\| + \|T^x_\theta(\theta) - T^x_\theta(\theta')\|$$

$$\leq p \|x^*_\theta - x^*_\theta'\| + \|T^x_\theta(\theta) - T^x_\theta(\theta')\|$$

$$\leq $$
\[ \Rightarrow \| x^*_0 - x^*_0 \| \leq \frac{1}{1 - \rho} \| x^* T(\theta) - x^* T(\theta') \| < \frac{\epsilon}{1 - \rho} \text{ whenever } d(\theta, \theta') < \frac{\epsilon}{x_{1,0}} \]

Example 1 (Jacobi’s algorithm)

Consider the system of linear equations

\[ Ax = b , \]

where \( A \) is a square matrix. We can write this as a fixed point problem

\[ x = -D^{-1}(L+U)x + D^{-1}b \]

where \( A = L + D + U \) denotes the decomposition of \( A \) into strictly lower triangular, diagonal and strictly upper triangular parts and we assume \( D \) to be invertible.

Jacobi’s algorithm to solve this problem:

\[ x_{k+1} = -D^{-1}(L+U)x_k + D^{-1}b \]

is a special case of Banach iteration for the map \( T(x) = -D^{-1}(L+U)x + D^{-1}b \)

Suppose \( A \) is diagonally dominant:

\[ |a_{ii}| > \sum_{j=1, j \neq i}^{n} |a_{ij}| \quad x = 1, 2, \ldots, n \]
Then \( T : \mathbb{R}^n \to \mathbb{R}^n \) is a contraction.

What is \( f \) in this case?

**Example 2** Consider the scalar equation

\[ g(x) = x^2 - b \text{ where } b > 0. \]

Let \( y = 1 - x \). Finding the (positive) square root of \( b \) is a fixed point problem,

\[ y = \frac{1}{2} (1 - b + y^2) = T(y). \]

Suppose \( |1 - b| < f < 1 \).

Then \( T \) maps the closed subset

\[ S = \{ y : |y| \leq f \} \subset \mathbb{R} \]

into itself and it is a contraction on \( S \) with parameter \( f \).

Thus the algorithm (Bézout iteration)

\[ y_{n+1} = \frac{1}{2} ((1 - b) + y_n^2) \]

converges for \( |1 - b| < f < 1 \)

It is equivalent to

\[ x_{n+1} = x_n - \frac{1}{2} x_n^2 + \frac{1}{2} b \]

How does it compare with Newton’s method?