Consider the functional
\[ J[x] = \int_{t_1}^{t_2} L(t, x(t), \dot{x}(t)) \, dt \]
defined on \( D = \{ t \in [t_1, t_2] \} \) for \( L \) sufficiently differentiable.

Suppose \( x(t_1) = x_1 \) and \( x(t_2) = x_2 \)
are fixed. Define the variation of \( J \)
\[ J^\varepsilon [x] = \int_{t_1}^{t_2} L(t, x(t) + \varepsilon \phi(t), \dot{x}(t) + \varepsilon \phi'(t)) \, dt \]
under a variation \( \phi = \varepsilon \eta \) where
\[ \eta(t_1) = \eta(t_2) = 0. \]

\[ \frac{d}{d\varepsilon} \bigg|_{\varepsilon = 0} J^\varepsilon [x] = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial x} (t, x(t), \dot{x}(t)) \eta(t) + \frac{\partial L}{\partial \dot{x}} (t, x(t), \dot{x}(t)) \eta'(t) \right) dt \]
and
\[ \frac{d^2}{d\varepsilon^2} \bigg|_{\varepsilon = 0} J^\varepsilon [x] = \int_{t_1}^{t_2} \left[ \frac{\partial^2 L}{\partial x^2} (t, x(t), \dot{x}(t)) \eta(t)^2 + 2 \frac{\partial^2 L}{\partial x \partial \dot{x}} (t, x(t), \dot{x}(t)) \eta(t) \eta'(t) \right. \]
\[ + \left. \frac{\partial^2 L}{\partial \dot{x}^2} (t, x(t), \dot{x}(t)) \eta'(t)^2 \right] dt \]
We would like to refer to \( \frac{d^2 J}{d^2 \varepsilon^2} \) above as the second variation denoted by

\[
\delta^2 J[h] = \int_{t_1}^{t_2} \left[ \frac{\partial^2 L}{\partial x \partial x} \dot{x}^2 + 2 \frac{\partial L}{\partial x} \dot{x} \dot{\varepsilon} + \frac{\partial^2 L}{\partial \varepsilon^2} \varepsilon^2 \right] dt
\]

We can get rid of the \( \varepsilon^2 \) term in the integral by integration by parts (we have assumed sufficient differentiability). Observe

\[
\int_{t_1}^{t_2} 2 \frac{\partial L}{\partial x} \dot{x} \dot{\varepsilon} dt
\]

\[
= \int_{t_1}^{t_2} \frac{\partial L}{\partial x} \frac{d}{dt} \left( \frac{1}{2} \dot{\varepsilon}^2 \right) dt
\]

\[
= -\int_{t_1}^{t_2} \frac{d}{dt} \left( \frac{\partial L}{\partial x} \dot{x} \right) \dot{\varepsilon} dt
\]

\[
= -\int_{t_1}^{t_2} \frac{\partial^2 L}{\partial x \partial \dot{x}} \varepsilon dt
\]

\[
= -\int_{t_1}^{t_2} \frac{\partial^2 L}{\partial x \partial \dot{x}} \dot{x}^2 dt.
\]

Hence

\[
\delta^2 J[h] = \int_{t_1}^{t_2} \left[ P(t) \dot{x}(t) \dot{\varepsilon} + Q(t) \varepsilon^2 \right] dt,
\]

where

\[
P(t) = \frac{\partial^2 L}{\partial x \partial \dot{x}} (t, x(t), \dot{x}(t) ; Q(t) = \frac{\partial^2 L}{\partial \varepsilon^2} (t, x(t), x(t)) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} (t, x(t), x(t)) \right)
\]
There (second order necessary conditions).

Suppose $J[x]$ attains a local minimum at $x$ ($\Rightarrow J^2[x]$ attains a local minimum at $x=0$). Then:

(a) $\frac{dJ^2}{de}igg|_{e=0} = 0 \iff E-L$ holds

(b) $\frac{d^2J^2}{de^2} \bigg|_{e=0} > 0 \Rightarrow$ Legendre

(Legendre) $\frac{\partial^2 L}{\partial x \partial \dot{x}} (t, x(t), \dot{x}(t)) > 0$

at each $t$.

Proof
(a) is already known

(b) That $\frac{d^2J^2}{de^2} > 0$ is a special case of Theorem on necessary conditions in lecture 11(a) page 1.

The Legendre condition follows from this. To see this suppose not: say

$P(t) = \frac{\partial^2 L}{\partial x \partial \dot{x}} (t, x(t), \dot{x}(t)) = -2\beta, \quad \beta > 0$

for some $t \in [t_1, t_2]$. 

\[ \beta \]
By continuity of $P(t)$, there exists an $\alpha > 0$

such that $t_1 \leq t_0 - \alpha$, $t_0 + \alpha \leq t_2$, and

$$P(t) < -\beta \quad t_0 - \alpha \leq t \leq t_0 + \alpha$$

Consider

$$k(t) \in \mathbb{D} (t, t_2)$$

such that

$$k(t) = \begin{cases} \sin^2 \frac{\pi (t-t_0)}{\alpha} & t_0 - \alpha \leq t \leq t_0 + \alpha \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\int_{t_1}^{t_2} (P(t) \, k(t)^2 + Q(t) \, k(t)^2) \, dt$$

$$= \int_{t_0 - \alpha}^{t_0 + \alpha} P(t) \, \frac{\pi^2}{\alpha^2} \sin^4 \frac{2\pi (t-t_0)}{\alpha} \, dt$$

$$+ \int_{t_0 - \alpha}^{t_0 + \alpha} Q(t) \, \sin^4 \frac{\pi (t-t_0)}{\alpha} \, dt$$

$$\leq -\beta \frac{\pi^2}{\alpha^2} 2\alpha + 2M \gamma$$

(Where $M = \max_{[t_1, t_2]} |Q(t)|$)

for sufficiently small $\alpha$ the r.h.s. above becomes negative which is a contradiction.

Thus $P(t) > 0$ for $t \in [t_1, t_2]$ (hence $\alpha$ is unnecessary)

\[\square\]
depend, we successfully attempted to turn this into a sufficient condition by the strengthened condition

\[ P(c) > 0 \]

(analogous to argument in Lecture 11(a) page 2), and the following sophisticated completion of square trick.

Observe, for an arbitrary differentiable \( w(t) \),

\[
0 = \int_{t_1}^{t_2} \frac{d}{dt} \left( w(t) h^2(t) \right) dt
\]

\[
= \int_{t_1}^{t_2} \left( w h^2(t) + 2w h(t) h'(t) \right) dt
\]

Adding this to \( S^2 J[h] \) we get

\[
S^2 J[h] = \int_{t_1}^{t_2} \left( P(c) h(t) + Q(c) h^2(t) \right) dt
\]

\[
= \int_{t_1}^{t_2} \left( P(c) h(t) + 2w h(t) h'(t) \right) dt
\]

\[
+ \left( Q(c) + \dot{w}(t) h(t) \right) \left( h(t) \right)^2 dt
\]
\[
\int_{t_1}^{t_2} \left[ \sqrt{P(t) + (Q(t) + \pi(t))} h(t) \right]^2 dt
\]

If \( \pi \) could be found then \( \delta J[R] > 0 \)
\( \neq 0 \), and sufficiency applies.

The catch is that the Riccati equation
\[
P(t) (Q(t) + \pi(t)) = \pi^2(t)
\]
need not have a solution \( \pi(t) \) for the entire interval \([t_1, t_2]\). Finite escape time for Riccati equations creates a problem. We need

**Definition (Conjugate Points):**

A point \( \bar{t} \neq t_1, t_1 < \bar{t} \leq t_2 \)
is said to be conjugate to \( t_1 \), if the
equation,
\[ \frac{d}{dt} (P(t) \dot{h}(t)) = Q(t) h(t) \]

has a solution which vanishes at \( t = t_1 \) and in the interval \( [t_1, t_2] \) but is not identically zero on \( [t_1, t_2] \).

Remark. The above differential equation is simply the Euler-Lagrange equation for the quadratic functional
\[ \int_{t_1}^{t_2} \left[ P(t) \dot{h}(t)^2 + Q(t) h(t)^2 \right] dt. \]

Theorem. If \( P(t) > 0 \) on \( [t_1, t_2] \) and if the interval \( [t_1, t_2] \) contains no points conjugate to \( t_1 \), then the quadratic functional
\[ \int_{t_1}^{t_2} \left[ P(t) \dot{h}(t)^2 + Q(t) h(t)^2 \right] dt \]
is positive definite for all \( h(t) \) s.t.
\[ h(t_1) = h(t_2) = 0. \]
\textbf{Proof}

Consider the equation

$$d(P(t) u(t)) = Q(t) u(t).$$

In the absence of conjugate points to \( t \), (by hypothesis), this equation has a solution \( u(t) \) which does not vanish anywhere on the interval \( [t_1, t_2] \).

Let \( w(t) = -\frac{u(t) P(t)}{u(t)} \).

Then

$$w = \frac{\dot{u}}{u^2} + \frac{1}{u^2} \dot{u} u P(t)$$

$$= -\frac{1}{u} \frac{d}{dt} (u(t) P(t))$$

Then

$$P(t) (Q(t) + w(t))$$

$$= P(t) \left( \frac{1}{u^2} \dot{u}^2 P(t) = \frac{1}{u} \frac{d}{dt} (u(t) P(t)) + Q(t) \right)$$

$$= \frac{\dot{u}^2(t) P(t)}{u(t)} - \frac{P(t)}{u(t)} \left\{ \frac{d}{dt} (u(t) P(t)) - Q(t) u(t) \right\}$$

$$= \dot{w}(t) - 0$$

Thus we can write,

$$-8-$$
\[\begin{align*}
&= \int_{t_1}^{t_2} \left( (p(t) \dot{\varphi}(t))^2 + 2w(t) \varphi(t) \dot{\varphi}(t) + \left(2w(t) \varphi(t) \right)^2 \right) dt \\
&= \int_{t_1}^{t_2} p(t) \left( \dot{\varphi}(t) + \frac{w(t) \varphi(t)}{p(t)} \right)^2 dt \\
&= 0 \quad (h(t) \neq 0) \\
\end{align*}\]

Suppose that expression vanishes for some \(h\).

Then \(\dot{h}(t) + w(t) \frac{h(t)}{p(t)} = 0\) \(t \in [t_1, t_2]\).

Setting \(h(t_1) = 0\) we find that \(h(t) = 0\).

By uniqueness of solution to ode's

\(h(t) \equiv 0\).

\[\Rightarrow\text{ Positive definiteness.}\]