Consider the problem of minimizing
\[
\eta = \int_{t_0}^{t_1} \left( x'(t) L(t) x(t) + u'(t) u(t) \right) dt + x'(t_1) Q x(t_1)
\]
along trajectories of
\[
\dot{x}(t) = A(t) x(t) + B(t) u(t)
\]
and \( x(t_0) = x_0 \). Without loss of generality assume that \( L(t) = L'(t) \) and \( A = A' \).

\( \eta \) contains a trajectory cost, a control cost and a terminal cost.

One can find an optimal control \( u^*_0 \) for this free end-point problem by a completion of square, trick. This involves the Riccati equation
\[
\dot{K}(t) = -A(t) K(t) - K(t) A(t) + K(t) B(t) B'(t) K(t) - L(t)
\]

Everything we do hinges on the Fundamental Lemma (path independent integrals)

Given \((x, u)\) satisfying
\[
\dot{x}(t) = A(t) x(t) + B(t) u(t),
\]
\[
\dot{x}(t_1) K(t_1) x(t_1) = x'(t_0) K(t_0) x(t_0)
\]
\[
= \int_{t_0}^{t_1} \begin{bmatrix} 0 & B(t) K(t) \\ K(t) B(t) & K(t) + A'(t) K(t) + K(t) A(t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt
\]

Proof: For the statement of the lemma to make
sense, we only need $K(t)$ to be any symmetric matrix-valued function on $[t_0, t_1]$ with a continuous derivative $K$ on $(t_0, t_1)$.

\[
\text{l.h.s.} = \int_{t_0}^{t_1} \frac{d}{dt} \left( x'(t) K(t) x(t) \right) \, dt
\]

\[
= \int_{t_0}^{t_1} (x'Kx + x'Kx + x'Kx) \, dt
\]

\[
= \int_{t_0}^{t_1} (Ax + Bu)'Kx + x'Kx + x'K(Ax + Bu) \, dt
\]

\[
= \text{r.h.s.}
\]

\[\square\]

Remark: We use it below in the form

\[
0 = \int_{t_0}^{t_1} \left( (u', x') \begin{pmatrix} 0 & BK \\ KB & K + A'K + KA \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix} \right) \, dt
\]

\[
+ x(t_0) K(t_0) x(t_0) - x(t_1) K(t_1) x(t_1)
\]

and we add this to "anything & anything."

**Theorem 1** Let $A(\cdot)$, $B(\cdot)$, $L(\cdot) = L'(\cdot)$ and $Q = A' \otimes A$ be given. Suppose that there exists a solution $\Pi = \Pi(t, x, t_1)$ of the Riccati equation

\[
K = -A'K - KA + KBB'K - L
\]

$K(t_1) = Q$.

Then there exists a control $u$ which minimizes

\[
\rho = \int_{t_0}^{t_1} \left[ u'(t) u(t) + x'(t) L(t) x(t) \right] \, dt + x(t_1) Q x(t_1)
\]
for the system \( \dot{x}(t) = A(t)x(t) + B(t)u(t) \); \( x(t_0) = x_0 \).

The minimum value of \( \eta \) is given by

\[ x_0' T(t_0, Q, t_1) x_0 \]

The minimizing control in closed loop form is

\[ u_0(t) = -B(t) T(t, Q, t_1) x(t) \]

In open loop form

\[ u_0(t) = -B'(t) T(t, Q, t_1) \tilde{X}(t, t_0) x_0 \]

where \( \tilde{X}(t, t_0) \) is the transition matrix for the system

\[ \dot{x}(t) = (A(t) - B(t) B'(t) T(t, Q, t_1)) x(t) \]

Proof: From the fundamental lemma,

\[
\begin{align*}
\eta &= \eta + 0 \\
&= \int_{t_0}^{t_1} (u' x') \begin{pmatrix} A & 0 \\ 0 & L \end{pmatrix} (u) \ dt + \pi'(t_1) Q x(t_1) \\
&\quad + x_0' T(t_0, Q, t_1) x_0 - x'(t_1) Q x(t_1) \\
&\quad + \int_{t_0}^{t_1} (u' x') \begin{pmatrix} 0 & B'T' \\ -B'T' & T'B'T' + A'T'T + A'A \end{pmatrix} (x) \ dt \\
&= \int_{t_0}^{t_1} (u' x') \begin{pmatrix} 0 & B'T' \\ -B'T' & T'B'T' \end{pmatrix} (x) \ dt + x_0' T(t_0, Q, t_1) x_0 \\
&= \int_{t_0}^{t_1} \| u + B'T' x \|^2 \ dt + x_0' T(t_0, Q, t_1) x_0
\end{align*}
\]

Clearly, for the choice \( u = u_0 = -B'T' x \), \( \eta_{\text{min}} = x_0' T(t_0, Q, t_1) x_0 \).
Remark: here \( \| \cdot \| \) stands for the Euclidean norm
\[
\| x \| = \sqrt{x'x}
\]

**Theorem 2** (special case of \( L=0 \))

Let \( A(t), B(t), Q=Q' \) be given. Let \( W \) be the accessibility Gramian for the system
\[
\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0.
\]

If the matrix
\[
H(t, t_1) = W(t, t_1) + \Xi(t, t_1)Q^{-1}\Xi(t, t_1)
\]
is invertible on \([t_0, t_1]\), then there exists a control which minimizes
\[
M = \int_{t_0}^{t_1} u(t)u(t)\,dt + x'(t_1)Q^{-1}x(t_1),
\]
and it takes the form
\[
u(t) = -B'(t)H^{-1}(t, t_1)x(t).
\]

**Proof:** Set \( L=0 \) in Theorem 1 and the Riccati equation becomes
\[
\dot{K} = -A'K - KA + KBBK; \quad K(t_1) = Q
\]

\[M = K^{-1} \]
satisfies
\[
\dot{M} = MA' + AM - BB' \quad \text{(Lyapunov equation)}
\]

which is also satisfied by \( W(t, t_1) \) (see lecture 1 notes, page 11), and has the solution
\[
H(t, t_1) = [W(t, t_1) + \Xi(t, t_1)Q^{-1}\Xi(t, t_1)].
\]

(see matrix version of constants formulae).

- \( \Xi \) -
Then if $H^{-1}$ exists, it satisfies the Riccati equations and we apply Theorem 1 to get the rest. 

and the boundary condition $x(t_1) = H(t, t_1)^{-1}$ 

$\Rightarrow (Q^{-1})^{-1} = Q$ 

1.6 Fixed end-point problems

Theorem 3 Assume that there exists a symmetric matrix $K$, such that the solution $T(t, t_0, K, t_1)$ of the matrix Riccati equation

$$K = -A^T K - KA + KBK^T K - L$$

exists on $[t_0, t_1]$. Then a differentiable trajectory $x(t)$ on $[t_0, t_1]$ of the system

$$x(t) = A(t)x(t) + B(t)u(t); \quad x(t_0) = x_0,$$

and

$$x(t_1) = x_1,$$

minimizes

$$\eta = \int_{t_0}^{t_1} \left[u(t)x(t) + x(t)L(t)x(t)\right]dt$$

if and only if it minimizes

$$\eta_1 = \int_{t_0}^{t_1} v(t)v(t)dt$$

for the system

$$x(t) = (A(t) - B(t)B^T(t)T(t, K, t_1))x(t) + B(t)v(t)$$

$x(t_0) = x_0$; and $x(t_1) = x_1$. Moreover along any trajectory satisfying the boundary conditions

$$\eta = \eta_1 + x_0^T T(t_0, K, t_1)x_0 - x_1^T Kx_1.$$
Proof. From the fundamental lemma on path independent integrals,

\[ \eta = t_1 (u_1, x_1) \left[ \begin{array}{ccc} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & \Pi \end{array} \right] \begin{pmatrix} u_1 \\ v_1 \\ x_1 \end{pmatrix} dt \]

\[ + \int_{t_0}^{t_1} \frac{\partial}{\partial x'} \left( \begin{array}{ccc} 0 & B' \Pi & 0 \\ \Pi B & \Pi B B' \Pi & 0 \\ 0 & 0 & \Pi \end{array} \right) \begin{pmatrix} u_1 \\ v_1 \\ x_1 \end{pmatrix} dt \]

\[ + x_0' \Pi(t_0, k_1, t_1) x_0 - x_1' k_1 x_1 \]

\[ = \int_{t_0}^{t_1} \left[ \begin{array}{ccc} M & B' \Pi & 0 \\ \Pi B & \Pi B B' \Pi & 0 \\ 0 & 0 & \Pi \end{array} \right] \begin{pmatrix} u_1 \\ v_1 \\ x_1 \end{pmatrix} dt + x_0' \Pi(t_0, k_1, t_1) x_0 \]

\[ - x_1' k_1 x_1 \]

Let \( u(t) + B(t) \Pi(t, k_1, t_1) x(t) = V(t) \)

\[ \Rightarrow \eta = \eta_1 + x_0' \Pi(t_0, k_1, t_1) x_0 - x_1' k_1 x_1 \]

To minimize \( \eta \) we have to minimize \( \eta_1 \) (w.r.t. \( V(t) \))

This completes the proof. \( \square \)

Remark. \( V(t) = 0 \) does not work in general since it may not get us to \((x_1, t_1)\).

So how do we compute optimal \( V(t) \)?
From Lecture 4, page 8,

\[ v_0(t) = -B'(t) \bar{E}'(t_0, t) \delta \]

where \( \delta \) is such that

\[ \bar{W}(t_0, t_1) \delta = x_0 - \frac{\bar{E}'(t_0, t_1) x_1}{A - BB'_{11}} \]

and

\[ \bar{W}(t_0, t_1) \] is the accessibility Gramian for the system

\[ \dot{x}(t) = (A(t) - B(t)B'(t)TT(t, k_i, t_i))x(t) \]

\[ + B(t)v(t) \]

and

\[ HH \]

is the corresponding transition matrix.

Then

\[ u_0(t) = v_0(t) - B(t)TT(t, k_i, t_i)x(t) \]

(open loop term)

(feedback term)

This theorem shows the structure of solution to a fixed end-point optimal control problem. It takes the form of a sum of an open loop part and a feedback part.
Given \((x_0, t_0)\) and \((x_1, t_1)\) we can

(i) solve \(T(t, \tau, K_1, t_1)\)

(ii) precompute \(v_0(t)\)

(iii) precompute the gain matrix \(-B(t)T(t, K_1, t_1)\)

Step (ii) requires us to compute the Gramian

\[
W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, s) B(s) B(s)^T \Phi(s, t_0) ds \quad A - BB'\Pi
\]

which in turn requires us to compute the transition matrix \(\Phi(\cdot, t_0)\) which satisfies

\[
\frac{d}{dt} \Phi(t, t_0) = (A(t) - BB'T(t, K_1, t_1)) \Phi(t, t_0) A - BB'\Pi
\]

Even if the original system is time invariant, this last step involves computing the transition matrix of a time-varying system, due to the time dependence of \(T(t, K_1, t_1)\). This is tedious!