from the Calculus of Variations.

The E-L equations and the Legendre condition can be interpreted in the following way via the control Hamiltonian $H$. As different from mechanics,

Define $H = H(t, q, p, u)$

to be a function of time, state, costate, and control

as

$$H(t, q, p, u) = p f(t, q, u) - L(t, q, u)$$

associated to the problem of optimal control,

$$\min_{u(t)} \int_{t_1}^{t_2} L(t, q(t), u(t)) \, dt$$

Subject to $q(t) = f(t, q(t), u(t))$

Boundary condition $q(t_1) = q_1, \quad q(t_2) = q_2$

In the special case $f(t, q, u) = u$, the optimal control problem is expressible as a problem of the calculus of variations and then the necessary condition is
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}(t)} - \frac{\partial L}{\partial q(t)} = 0 \]  
\[ (E-L) \]

\[ \frac{\partial^2 L}{\partial u^2} (t, q(t), \dot{q}(t)) \geq 0 \]  
\[ \text{Legendre} \]

The control Hamiltonian in the special case is:

\[ H^H(t, q, p, u) \]

\[ = p u - L(t, q, u) \]

Observe:

\[ \frac{\partial H^H}{\partial p} = u \]

\[ \frac{\partial H^H}{\partial q} = -\frac{\partial L}{\partial q} \]

\[ \frac{\partial H^H}{\partial \dot{q}} = p - \frac{\partial L}{\partial u} \]

\[ \frac{\partial^2 H^H}{\partial u^2} = -\frac{\partial^2 L}{\partial u^2} \]

Now, suppose \( t \mapsto q(t) \) is a trajectory of the system \( \dot{q}(t) = u(t) \) which satisfies the boundary conditions and minimizes \( t_2 \)

\[ \int_{t_1}^{t_2} L(t, q(t), u(t)) \, dt \]
\[ t \mapsto p(t) = \frac{\partial L(t, q(t), \dot{q}(t))}{\partial q} \]  

Then by E-L,

\[ \dot{q}(t) = u(t) = \frac{\partial H}{\partial p} (t, q(t), p(t), \dot{q}(t)) \]

\[ \dot{p}(t) = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}(t, q(t), \dot{q}(t)) \]

\[(C) \]

\[ = \frac{\partial L}{\partial q} (t, q(t), \dot{q}(t)) \]

\[ = -\frac{\partial H}{\partial q} (t, q(t), p(t), \dot{q}(t)) ; \]

by definition of \( t \mapsto p(t) \)

\[(M1) \quad \frac{\partial H}{\partial u}(t, q(t), p(t), \dot{q}(t)) = 0 \]

by Legendre

\[(M2) \quad \frac{\partial^2 H}{\partial u^2} (t, q(t), p(t), \dot{q}(t)) \leq 0 \]

Conditioning \( (M1) \) and \( (M2) \) are necessary for

\[ \| -H(t, q(t), p(t), \dot{q}(t)) \|

\[ = \max_{u} \| -H(t, q(t), p(t), \dot{q}(t)) \| u \]

\[ -3 \]
This suggests a conjecture / principle.

Consider the optimal control problem

\[
\begin{align*}
\min_{u(t)} & \quad \int_{t_1}^{t_2} L(t, q(t), u(t)) \, dt \\
\text{subject to} & \quad \dot{q}(t) = f(t, q(t), u(t)) \\
\text{BC} & \quad q(t_1) = q_1, \quad q(t_2) = q_2 \\
\end{align*}
\]

Associate the Hamiltonian

\[
H = H_1(t, q, p, u) = p f(t, q, u) - L(t, q, u).
\]

Suppose the input-state pair

\[ t \mapsto (q^*(t), u^*(t)) \]

solves the optimal control problem.

Then there exists a co-state trajectory

\[ t \mapsto p(t) \] such that

\[ \dot{q}^*(t) = \frac{\partial H}{\partial p} (t, q^*(t), p(t), u^*(t)) \]

and

\[ \dot{p}(t) = -\frac{\partial H}{\partial q} (t, q^*(t), p(t), u^*(t)) \]

and

\[ H(t, q^*(t), p^*(t), u^*(t)) = \max_u H(t, q^*, p, u) \]

\[ - \rightarrow \]
\[ H(t, q^*(t), p(t), u^*(t)) \]

\[ = \max_u H(t, q^*(t), p(t), u) \]

This is, essentially, modulo some technicalities on the space of control functions, the co-workers).

See your own class notes for details of examples.

The statement above extends to multidimensional systems by letting

\[ H(t, q, p, u) \]

\[ = p^T f(t, q, u) - L(t, q, u) \]