1. In Lecture Notes 8, page 4, example 1, compute $p$.

2. In Lecture Notes 8, page 5, example 2, do a numerical comparison (using MATLAB) of the stated algorithm with Newton's algorithm, for several values of $b > 0$. Explain and discuss your data.

3. The second Fréchet derivative of $T: X \rightarrow Y$ is $D^2 T(x)$ defined by

$$D^2 T(x)(h,k) = \frac{d^2 T(x + th + sk)}{ds dt}_{s=0, t=0}$$

where $h, k \in X$.

$D^2 T(x)(h,h)$ is called the second variation of $T$ with increment $h$.

For the function $J[z] = \int_{t_1}^{t_2} L(t, x(t), x'(t)) dt$ defined on differentiable curves $x(t)$ with fixed end points $x(t_1) = x_1$ and $x(t_2) = x_2$, show that the second variation can be written as

$$D^2 J[z](h,h) = \int_{t_1}^{t_2} \left( P(t) h(t)^2 + Q(t) h'(t)^2 \right) dt.$$
4. Let $X$ be a Banach space. Let $A : X \to X$ be a bounded linear operator. Suppose $\|A\| = a < 1$. Use the contraction mapping theorem to show that,

$$
(1 - A) \text{ is invertible, and,}
$$

$$
\| (1 - A)^{-1} \| < \frac{1}{1 - a}.
$$

5. In problem 2 above, replace the Newton algorithm by

(i) The modified Newton algorithm

$$
x_{n+1} = x_n - \lambda_n (D P(x_n))^{-1} P(x_n)
$$

where $\lambda_n$ is a step-size parameter selected according to the Armijo step-size rule (This is the Armijo-Newton method — see page 55 of Tits’ notes and page 37 of Sameh notes).

(ii) Carry out a numerical comparison with the results of problem 2. Which of the three algorithms is empirically the fastest?