Consider \( \dot{x} = f(x) \). Let \( \Phi^f_t : \mathbb{R}^n \to \mathbb{R}^n \) denote the associated flow map, i.e. 
\[
\Phi^f_t(x_0) = \text{solution to the differential equation } \dot{x} = f(x) \text{ evaluated at time } t, \text{ with initial condition } x(t=0) = x_0.
\]

Here without further elaboration we assume that the given ODE satisfies conditions for existence and uniqueness, so that for (open) domain \( D \subseteq \mathbb{R}^n \),
\[
\Phi^f_t : D \to \mathbb{R}^n
\]
is smooth and has smooth inverse on \( \Phi^f_t(D) \).

Recall that if \( Y \) is an \( m \)-dimensional random variable \( Y = \Phi^f(x) = \text{image under a smooth and smoothly invertible transformation } \Phi : \mathbb{R}^n \to \mathbb{R}^m \) of another \( m \)-dimensional random variable \( X \), with density \( p_x \), then \( Y \) has density
\[
p_Y(y) = p_x(\Phi^{-1}(y)) \left| \det \left( \frac{\partial \Phi^{-1}(y)}{\partial y} \right) \right|^{-1} \quad \text{for } y \in \Phi(X)
\]
(this is the "change of variables formula").

Suppose the initial condition for \( \dot{x} = f(x) \) is random with density \( p_d(x) \). Using the change of variables formula above we
seek to determine how the density evolves. Clearly

\[ p(t, \Phi_t^F(x)) \cdot \det \left( \frac{\partial \Phi_t^F}{\partial x} \right) = p_0(x) \]

Differentiate both sides to obtain

\[ 0 = \frac{1}{\partial t} \left( p(t, \Phi_t^F(x)) \cdot \det \left( \frac{\partial \Phi_t^F}{\partial x} \right) \right) \]

\[ = \left( \frac{\partial p}{\partial t} + \sum_{i=1}^{n} \frac{\partial p}{\partial x_i} \cdot \frac{1}{\partial t} \left( \Phi_t^F(x_i) \right) \right) \cdot \det \left( \frac{\partial \Phi_t^F}{\partial x} \right) \]

\[ + p \cdot \frac{1}{\partial t} \left( \frac{\partial \Phi_t^F}{\partial x} \right) \]  
(Chain Rule)

\[ = \left( \frac{\partial p}{\partial t} + \sum_{i=1}^{n} \frac{\partial p}{\partial x_i} \cdot f(\Phi_t^F(x)) \right) \cdot \det \left( \frac{\partial \Phi_t^F}{\partial x} \right) \]

\[ + p \frac{1}{\partial t} \left( \det \left( \frac{\partial \Phi_t^F}{\partial x} \right) \right) \]  
(Definition of Flow)

\[ = (I) + (II) \]

In (I), \( p = p(t, \Phi_t^F(x)) \) and the arguments for partial derivatives are as for \( p \).
To evaluate (3), let us denote
\[ g(t) = \frac{\partial \Phi_t}{\partial x} \]
(suppressing the dependence on \( x \) to avoid clutter),
\[ \det(g) = \sum_{j=1}^{n} g_{ij} C_{ij} \]
for any \( i \in \{1, 2, \ldots, n\} \). Here \( C_{ij} \) denote
the cofactor of the \( (i, j)^{th} \) element of
the matrix \( g \), i.e., \( (-1)^{i+j} \) the determinant
of the \((n-1) \times (n-1)\) submatrix of \( g \) obtained
by removing the \( i^{th} \) row and \( j^{th} \) column.
Clearly, \( C_{ij} \) does not include the variable \( g_{ij} \).

By chain rule,
\[ \frac{d}{dt} \det(g) = \sum_{i=1}^{n} \sum_{j=1}^{n} 2 \frac{\partial \det(g)}{\partial g_{ij}} \frac{d g_{ij}}{dt} \]
\[ = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} \frac{d g_{ij}}{dt} \quad \text{from (4)} \]

where \( C^T = [C_{ij}]^T = \text{transpose of matrix of cofactors} \).

On the other hand \( g^{-1} = \frac{C^T}{\det(g)} \),
equivalently \( C^T = \det(g) g^{-1} \).
Thus

\[
\frac{d}{dt} \det(g) = \text{tr} \left( \det(g) g^{-1} \dot{g} \right)
\]

\[
= \det(g) \cdot \text{tr} \left( g^{-1} \dot{g} \right)
\]

\[
= \det(g) \cdot \text{tr} \left( \dot{g} \dot{g}^{-1} \right)
\]

We compute,

\[
\dot{g} = \frac{d}{dt} \frac{\partial \Phi^f}{\partial x} = \left[ \frac{d}{dt} \frac{\partial \Phi^f}{\partial x} \right]
\]

\[
= \left[ \frac{\partial}{\partial x_j} \frac{d}{dt} \Phi^f_t \right]
\]

\[
= \left[ \frac{\partial}{\partial x_j} \left. \frac{d}{dt} \Phi^f_t (x) \right|_{x_0} \right]
\]

\[
= \left[ \frac{\partial}{\partial x_j} \frac{d}{dt} \Phi^f_t (x) \right]
\]

\[
\left( \text{CHAIN RULE} \right)
\]

\[
= \frac{\partial}{\partial x} \left. \frac{d}{dt} \Phi^f_t (x) \right|_{x_0}
\]

\[
= \frac{\partial}{\partial x} \left. \frac{d}{dt} \Phi^f_t (x) \right|_{x_0} g
\]

\[
(\text{5})
\]
Thus
\[
\frac{d}{dt} \det(g) = \det(g) \tr \left( g g^{-1} \right)
\]
\[= \det(g) \tr \left( \frac{\partial f(\Phi_t(x))}{\partial x} \right) \]

(6)
\[= \det \left( \sum_{i=1}^{\nu} \frac{\partial f}{\partial x_i} (\Phi_t(x)) \right) \sum_{i=1}^{\nu} \frac{\partial f_i}{\partial x_i} (\Phi_t(x)) \]

It follows that
\[0 = (I) + (II) \]
\[= \det \left( \frac{\partial \Phi_t(x)}{\partial x} \right) \cdot \left\{ \frac{\partial p}{\partial t} + \sum_{i=1}^{\nu} \left( \frac{\partial p}{\partial x_i} f_i + p \frac{\partial f_i}{\partial x_i} \right) \right\} \]

Since \( \det \left( \frac{\partial \Phi_t(x)}{\partial x} \right) \neq 0 \) (recall \( \Phi_t \) is invertible), it follows that
\[0 = \frac{\partial p}{\partial t} + \sum_{i=1}^{\nu} \frac{\partial (pf_i)}{\partial x_i} \]

(7)
\[= \frac{\partial p}{\partial t} + \text{div} (pf) \]

This is Liouville's theorem.
Suppose \( n = 2K \), \( x = (q_1, q_2, \ldots, q_K, p_1, p_2, \ldots, p_K) \),
\[
\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, 2, \ldots, K.
\]

Then
\[
f = \begin{pmatrix} \frac{\partial H}{\partial q} \\ -\frac{\partial H}{\partial q} \end{pmatrix} \quad \text{(Hamiltonian vector field)}
\]

Then \( 2K \)
\[
\text{div}(f) = \sum_{i=1}^{2K} \dot{q}_i = \sum_{i=1}^{K} \left( \frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} \right) = 0
\]

Hence
\[
0 = \frac{\partial p}{\partial t} + \text{div} (p f)
\]
\[
= \frac{\partial p}{\partial t} (t, q, p) + \sum_{i=1}^{K} \left( \frac{\partial p}{\partial q_i} \dot{q}_i + \frac{\partial p}{\partial p_i} \dot{p}_i \right)
\]
\[
= \frac{dp}{dt} \quad \text{along trajectories of}
\]
\[
x = f(x) \quad \text{Hamiltonian vector field}
\]

This is a conservation law \( \text{CMLBB5} \)
FLUIDS VIEW

of Liouville's Theorem

\[ \frac{d}{dt} m(W,t) = \frac{d}{dt} \int_W f(x,t) \, dv \]

W smooth, \( \partial W \) smooth
\( n \) outward normal vector

\[ \int_W \frac{\partial}{\partial t} f(x,t) \, dv \]

Volume flow rate across \( \partial W \)

\[ = \text{area element on } \partial W \]

Rate of increase of mass in \( W \)
rate at which mass is crossing the boundary \( \partial W \) inward direction

\[ \frac{d}{dt} \int_W f \, dv = - \int_{\partial W} \nabla f \cdot n \, dA \]
\[ \frac{d}{dt} \int_W f \ dv = - \int_W \text{div} (pu) \ dv \]

by divergence theorem.

\[ \Rightarrow \frac{d}{dt} \int_W \left( \frac{\partial p}{\partial t} + \text{div} (pu) \right) \ dv = 0 \]

Holds for all \( W \)

\[ \Rightarrow \frac{\partial p}{\partial t} + \text{div} (pu) = 0 \]