

The principles of mechanics as developed in the works of Laplace, Lagrange, and Dirichlet lead to tools for understanding the stability properties of solutions of nonlinear systems. The modern period in this direction begins with the classic memoir of A.M. Lyapunov (Problème générale de la stabilité de mouvement, Ann. Sci. Éc. Norm. Sup. (3) 1892, fascicule 9: 203-474 (1907) - translation of a paper in Russian published in Compt. Rend. Acad. Sci. Paris, 1892, fascicule reproduction as Annals of Mathematics Study No. 17 (Princeton Univ. Press, 1947)). The key ingredients are: (a) a definition of stability and, (b) an "energy method" to assess stability.

We introduce here in the setting of autonomous differential equations.

(1) An equilibrium point x_e of a system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ satisfies $f(x_e) = 0$. Let $y = x - x_e$ and define $g(y) = f(y + x_e)$. Then the equilibrium point $y = 0$ of the system $\dot{y} = g(y)$ corresponds to the equilibrium point x_e of the system (1), and any other trajectory $x(t)$ of (1) maps onto a trajectory $y(t) = x(t) - x_e$ of the system (2). Thus the shift of origin in \mathbb{R}^n allows me to refer to an equilibrium at 0. This is a standard device in most treatments - we don't use it below.

Lyapunov = Liapunoff = Liapunov

Definition (Lyapunov stability)

(i) The equilibrium x_e of (1) is said to be stable in the sense of Lyapunov, if given $\epsilon > 0$, there exists $\delta > 0$ such that $x_0 \in B_\delta(x_e)$ implies that

the solution starting at x_0 , denoted $x(t)$, is trapped in $B_\epsilon(x_e)$:

$$x(t) \in B_\epsilon(x_e) \quad \forall t \geq 0.$$

Here $B_r(z)$ stands for the open ball of radius r centered at z - $B_r(z) = \{x : \|x - z\| < r\}$, with respect to a choice of norm $\|\cdot\|$ in \mathbb{R}^n .

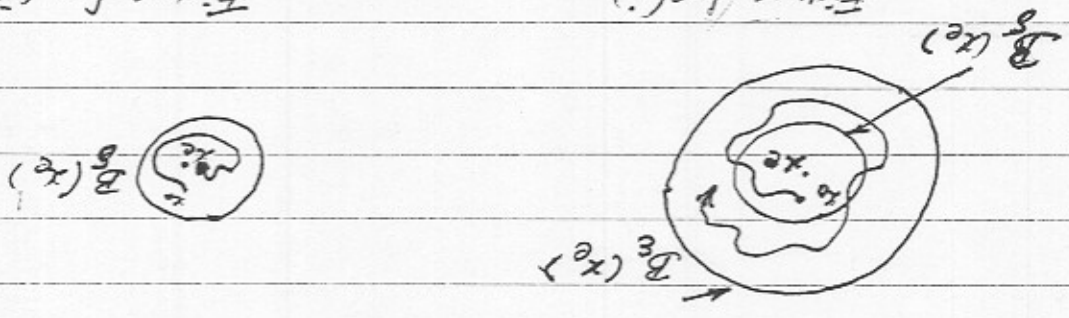


Figure for (i)

Figure for (iii)

(ii) x_e is unstable if not stable.

(iii) x_e is asymptotically stable if it is stable as in (i) and $\delta > 0$ can be

chosen such that $\lim_{t \rightarrow \infty} x(t) = x_e$ for

all initial condition $x_0 \in B_\delta(x_e)$ (attractivity of x_e)

The basic theorem of the subject, due to

Lyapunov gives a sufficient condition for stability (or asymptotic stability) of an

equilibrium point x_e .

Theorem (Lyapunov)

Let x_e be an equilibrium point of the system (satisfying local Lipschitz condition)

$$\dot{x} = f(x)$$

Let D be a domain (= open connected region) of \mathbb{R}^n containing x_e . Suppose $V: D \rightarrow \mathbb{R}$ is a C^1 function and

$$V(x_e) = 0$$

$$V(x) > 0 \text{ in } D - \{x_e\}$$

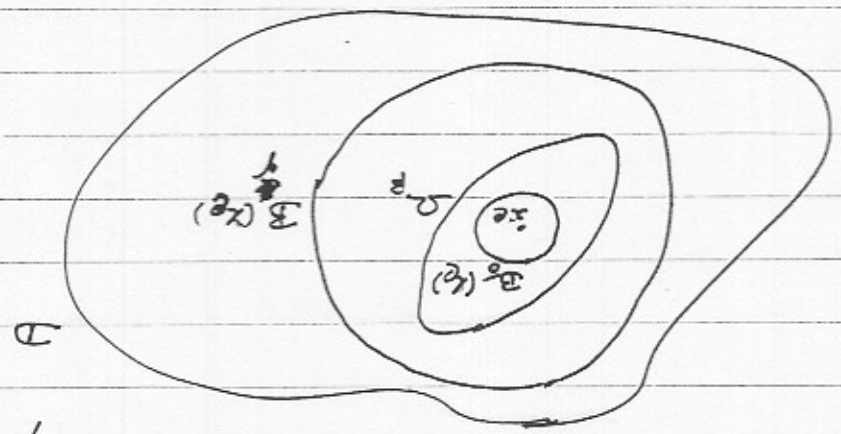
$$\dot{V}(x) \leq 0 \text{ in } D$$

Then x_e is stable. Moreover, if

$$\dot{V}(x) < 0 \text{ in } D - \{x_e\}$$

then x_e is asymptotically stable.

Proof The adjoining figure should be referred to in connection with the next lemma.



Given $\epsilon > 0$, choose $r \in (0, \epsilon]$ such that

$$\text{closed ball } \overline{B_r(x_e)} = \{x: \|x - x_e\| \leq r\} \subset D$$

Let $\alpha = \min_{\|x - x_e\| = r} V(x)$. Then $\alpha > 0$ by hypothesis.

Take β such that $0 < \beta < \alpha$, and let $\Omega_\beta = \{x \in \bar{B}_r(x_0) : V(x) \leq \beta\}$.

Then $\Omega_\beta \subset \bar{B}_r(x_0)$.

proof: suppose not, let $p \in \Omega_\beta$

be such that $\|p - x_0\| = r$.

Then $V(p) \geq \alpha > \beta$, a contradiction.

Let $x(t)$ be a solution with $x(0) \in \Omega_\beta$.

Since $V(x(t)) \leq 0$, $V(x(t)) \leq V(x(0))$ and

hence $x(t) \in \Omega_\beta$, $\forall t \geq 0$. Since Ω_β

is a closed and bounded set, we

conclude (see Theorem, Lecture Notes 4, page 16)

that we have a unique solution $c \in \Omega_\beta$,

a positively invariant set, for all time.

V is continuous at x_0 and $V(x_0) = 0$.

hence there is a $\delta > 0$ such that

$$\|x - x_0\| \leq \delta \Rightarrow V(x) < \beta.$$

Thus

$$\bar{B}_\delta(x_0) \subset \Omega_\beta \subset \bar{B}_r(x_0),$$

and

$$x(0) \in \bar{B}_\delta(x_0) \Rightarrow x(t) \in \Omega_\beta$$

$$\Rightarrow x(t) \in \Omega_\beta \quad \forall t \geq 0$$

$$\Rightarrow x(t) \in \bar{B}_r(x_0) \quad \forall t \geq 0$$

~~(contradiction)~~

(The $\delta > r$) $\Rightarrow x(t) \in \bar{B}_\delta(x_0) \quad \forall t \geq 0$

[These are an appeal to the 'well-ordering' of \mathbb{R}]

Under the extra assumption $V(0) < 0$

Let $D = \{x \in \mathbb{R} \mid \text{we can show that the}$

monotone decreasing function $V(x(t))$ has

a limit $c = 0$. The existence of a limit

$c \geq 0$ is ensured by the lower bound

on V . Suppose $c > 0$. By continuity of V

there is a $d > 0$ such that

$$B_d(x_c) \subset \Omega_c.$$

Since $\lim_{t \rightarrow \infty} V(x(t)) = c > 0$ by hypothesis,

$x(t)$ never enters $B_d(x_c)$.

Let $\gamma = \max_{x \in D} V(x)$

$$d \leq \|x - x_c\| < \gamma$$

By hypothesis, $\gamma > 0$. By the fundamental theorem

of integral calculus,

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau)) d\tau$$

$$\leq V(x(0)) - \gamma t$$

Since, the right hand side eventually becomes negative, we get a contradiction from

assuming $c > 0$. So $c = 0$. Hence

$$\lim_{t \rightarrow \infty} V(x(t)) = 0.$$

$t \rightarrow \infty$

By hypothesis on V , $\lim_{t \rightarrow 0} x(t) = x_0$

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Remark The above result is a prototype stability theorem in the spirit of the energy method in mechanics (more on this later). In the abstract setting of ordinary differential equations not necessarily Hamiltonian or dissipative, the Lyapunov function V is a stand-in for energy functions from mechanics.

Note that $v(x) = \frac{\partial v}{\partial x} \cdot f$. Suppose

the state space can be factored into position and momentum variables $x = (q, p) \in \mathbb{R}^n \times \mathbb{R}^n$ and

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned} \quad (3)$$

Then
$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial q} \cdot \frac{dq}{dt} + \frac{\partial H}{\partial p} \cdot \frac{dp}{dt} = 0$$

It we further assume that $x_e = (q_e, p_e)$ is an equilibrium with $H(x) \approx H(x_e)$ for all $x \in \mathcal{D}$ a neighborhood of x_e , then

$$V(x) = H(x) - H(x_e)$$

satisfies the hypotheses of Lyapunov's theorem allowing one to conclude x_e is a stable equilibrium.

Suppose $H(x)$ is of the form ~~kinetic energy~~ kinetic energy + potential energy.

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$$H(q, p) = \frac{1}{2} p \cdot M^{-1}(q) p + V(q)$$

where $H(q) = H(q) > 0$ is a map matrix and $V(q)$ is a potential, then (q_e, p_e) is an equilibrium of the dynamics

$$\begin{aligned} \dot{q} &= M^{-1} p \\ \dot{p} &= -\frac{\partial V}{\partial q} \end{aligned} \quad (A)$$

If $p = 0$ and q is a critical point of V then (q_e, p_e) is a strict local minimum of H and hence a stable equilibrium.

The result just derived is known as the Lagrange-Dirichlet theorem and is a guiding principle in much of mechanics. It asserts the proper role of (potential) energy minimization in stability in the context dynamical systems e.g. due to Torricelli.

Corollary If x_e is a (stable) equilibrium of a nontrivial Hamiltonian system, it can never be asymptotically stable.

Proof. Exce, $H(x_0) \neq H(x_e)$. For any trajectory beginning at x_0 , $H(x(t)) = H(x_0) \neq H(x_e)$. Hence, by continuity of H , $\lim_{t \rightarrow \infty} H(x(t)) \neq H(x_e)$ even if such a limit exists. \square

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Hamiltonian systems of the form (4) are said to be natural mechanical systems or rindple in classical physics, molecular dynamics, and engineering take this form, or its dissipative modification

$$\dot{q} = M^{-1}p$$

$(q, p) \in \mathbb{R}^{2n}$

$$(5) \quad \dot{p} = -\frac{\partial V}{\partial q} - R(q) \dot{q}$$

where $R(q) = R^T(q) > 0$, defines the Rayleigh dissipation function
 $R(q, \dot{q}) = \frac{1}{2} \dot{q}^T \cdot R(q) \dot{q}$

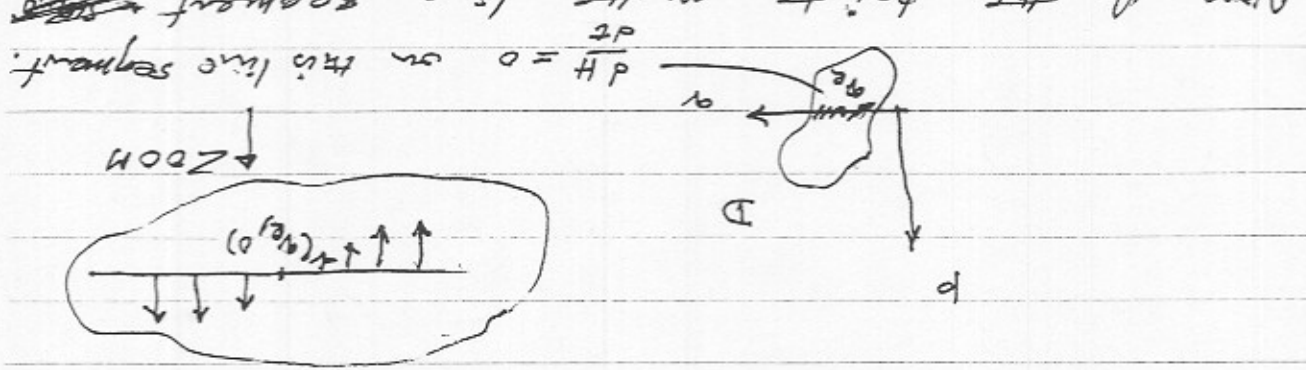
Along trajectories of (5),

$$\frac{dH}{dt} = -M^{-1}(q) \dot{q} \cdot (R(q) M^{-1}(q) \dot{q})$$

$$\frac{dH}{dt} \leq 0$$

by hypothesis

Suppose \mathcal{D} is such that $(q_e, 0)$ is the only equilibrium in \mathcal{D} (thus q_e is a critical point of V), and q_e is a strict local minimum of V . Then, can we prove asymptotic stability of $(q_e, 0)$? While stability is assured by Lyapunov-Dirichlet, one cannot use H as the Lyapunov function for the asymptotic stability argument. This is because $\frac{dH}{dt} \Big|_{(q, \dot{q})} = 0$



None of the points on the line segment ~~is~~ is an equilibrium (except for $(q_e, 0)$)

by hysteresis on D . Does this mean we get convergence to $(q_e, 0)$ anyway? One needs a new idea, - the invariance principle

of La. Salle, if one insists on working with H as a possible Lyapunov function.

An alternative is to fix up the Lyapunov function, i.e. add to H suitable extra terms. Both these approaches are important in solving a range of problems and we will discuss both.

Example (Fixing Lyapunov function) ~ damped pendulum
 In equation (5) consider $n=1$, $H = \frac{1}{2} \dot{\theta}^2 + V(\theta)$ constant,
 $V(\theta) = \frac{g}{2} (1 - \cos(\theta))$, $R(\theta) = b > 0$

Then $\dot{q} = p$
 $\dot{p} = -\frac{g}{2} \sin(\theta) - b p$
 $H(q, p) = \frac{1}{2} p^2 + \frac{g}{2} (1 - \cos(q))$
 $\frac{\partial H}{\partial p} = -b p^2 = 0$
 $(q, p) = (0, 0)$ are equilibria

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Define

$$\tilde{H} = \frac{z}{1} (q, p) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \frac{z}{g} (1 - \cos(q))$$

We seek $[a_{ij}] > 0$ such that

$$\frac{d\tilde{H}}{dt} = (a_{11}q + a_{12}p + \frac{z}{g} \sin(q)) p + (a_{12}q + a_{22}p - \frac{z}{g} \sin(q) - bp)$$

$$= \frac{z}{g} (1 - \frac{z}{g}) p \sin(q) - \frac{z}{g} a_{12} q \sin(q) + (a_{11} - a_{12}b) pq + (a_{12} - a_{22}b) p^2$$

< 0 on a suitable D .

Pick $a_{22} = 1$; $a_{11} = a_{12}b$ and $0 < a_{12} < b$. Then

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{12}b & a_{12} \\ a_{12} & 1 \end{pmatrix} > 0.$$

Further $\frac{d\tilde{H}}{dt} = -\frac{z}{g} a_{12} q \sin(q) - (b - a_{12}) p^2$

> 0 on $D = \{(q, p) \mid |q| < \pi\}$

Thus \tilde{H} is a Lyapunov function - $\{(0,0)\}$

ensuring asymptotic stability of $(0,0)$.

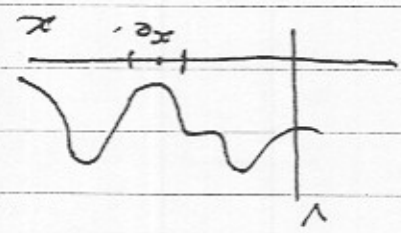
fixed term $H = H + \frac{z}{g} \sin^2(q) + a_{12} pq$

Example (Gradient Dynamics)

$V: \mathbb{R}^n \rightarrow \mathbb{R}$

$\dot{x} = -\nabla V(x)$

$x \in \mathbb{R}^n$



x_e is an equilibrium iff it is a critical point of V . Suppose it is an isolated local minimum of V . From $\frac{dV}{dt} = -\nabla V \cdot \nabla V = -\|\nabla V\|^2 < 0$

for all $x \in B_\epsilon(x_e) - \{x_e\}$ for $\epsilon > 0$ small enough we conclude asymptotic stability of x_e .

The invariance principle of LaSalle is based on a fundamental result of

G.D. Birkhoff, American mathematician of

the first half of the 20th century and a

1884-1944

founder of the modern theory of dynamical

systems.

Theorem (Birkhoff) If a trajectory $x(t), t \geq 0$ of a dynamical system is bounded, then $L_+ = \omega$ limit set of $\{x(t) : t \geq 0\}$ is a nonempty, compact, invariant set.

(on \mathbb{R}^n with metric $d(\cdot, \cdot)$, compact \iff closed and bounded)

Moreover $x(t) \rightarrow L_+$ as $t \rightarrow \infty$

in the sense that

$\lim_{t \rightarrow \infty} d(x(t), L_+) = 0$

note, for $x = f(x)$

$$V(x) = \frac{\partial V \cdot f(x)}{\partial x}$$

$$= \lim_{t \rightarrow \infty} \min_{p \in L_t} d(x(t), p) = 0.$$

we omit the proof of Birkhoff's theorem. (see for instance H. Khalil's appendix A.2)

Theorem (LaSalle)

Let Ω be a compact (closed and bounded) set with the property that $\forall t \geq 0, x(t) \in \Omega$, whenever $x(0) \in \Omega$.

Let $V: \Omega \rightarrow \mathbb{R}$ be a C^1 function

such that $V(x) \leq 0$ in Ω . Let E be the set of all points in Ω where $V(x) = 0$.

Let M be the ~~largest~~ largest invariant set in E . Then every solution starting in Ω tends to M as $t \rightarrow \infty$.

Proof: Let $x(t)$ be a solution such that $x(t) \in \Omega$

$t \geq 0$. Since $V(x) \leq 0$ in Ω , $V(x(t))$ is a

monotone decreasing function of t . Since $V(x(t))$

is continuous in the compact set Ω , it is bounded below on Ω . Therefore $V(x(t)) \rightarrow a$ as $t \rightarrow \infty$.

Let $C \subset \Omega$. Since Ω is closed, for any $p \in L_C$

there is a sequence $t_1 \leq t_2 \leq \dots \leq t_n \leq \dots$

such that $\lim_{n \rightarrow \infty} x(t_n) = p$.

By continuity of V , $V(p) = \lim_{n \rightarrow \infty} V(x(t_n)) = a$.

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Hence $V(x) = c$ on L_+ . Since L_+ is invariant (Dirichlet), $V(x) = 0$ on L_+ . Since M is the largest invariant set $c \in E$, we get $L_+ \subset M \subset E \subset \Omega$.
 Since $\{x(t)\}$ is bounded, $x(t) \rightarrow L_+$ as $t \rightarrow \infty$.
 Hence $x(t) \rightarrow M$ as $t \rightarrow \infty$ \square

Remark

In many problems M is much easier to determine than L_+ .

In some problems one can pick Ω and V such that, $x_e = \text{equilibrium} \in \Omega$ and in fact $M = \{x_e\}$. If further x_e is stable, it is asymptotically stable by LaSalle's theorem.

In (5),

$\Omega_c = \{(q, p) : H(q, p) \leq c\}$.
 For $c > 0$ such that $H(q_e, 0) \leq c$, c small enough, Ω_c is closed & bounded.
 $E = \{(q, p) : R(q)H(q) = 0\} \subset \Omega_c$

~~Therefore~~
 $M = \{(q_e, 0)\}$

proof \rightarrow next page.

$$E = \{ (q, p) \in \Omega_0 : p = 0 \}$$

$$(q(t), p(t)) \in M \quad \forall t \geq 0$$

$$\Rightarrow p(t) \equiv 0 \quad (M \subset E)$$

$$\Rightarrow p(t) \equiv 0$$

$$\Rightarrow (a) \quad q(t) \equiv 0 \quad (\text{from the dynamics})$$

$$\Rightarrow q(t) \equiv q_0 \text{ a constant}$$

$$\Rightarrow (b) \quad \frac{\partial V}{\partial v}(q(t)) \equiv 0 \quad (\text{again from the dynamics})$$

$$\Rightarrow \frac{\partial V}{\partial q}(q_0) = 0 \quad (\text{from (a) above})$$

Then we have shown that $(q, p) \in H$ implies $p = 0$ and q is a critical point of V . Choosing ϵ small enough we can insure $\{ (q_0, 0) \} = M$ where q_0 is a local minimum of V . Then we have shown that all trajectories starting in $\Omega_\epsilon \rightarrow (q_0, 0)$.