ENEE 661 Lecture 9 PSK 05.11.00

recall: indirect method ≡ stability assessment by looking at linearized dynamics (and eigenvalues of linearization if time invariant)

Center Manifold Theorem and Reduction

<based on H. Khalil – 2^{nd} edition>

The critical cases of Lyapunov, i.e. the systems for which the indirect method is inconclusive due to the presence of eigenvalues on the imaginary axis can be examined via the center manifold theorem.

Consider the nonlinear system

$$(*) \qquad \begin{aligned} \dot{y} &= A_1 y + g_1(y,z) \\ \dot{z} &= A_2 z + g_2(y,z) \end{aligned} \qquad \begin{aligned} A_1 &: \mathbb{R}^k \to \mathbb{R}^k \\ A_2 &: \mathbb{R}^{n-k} \to \mathbb{R}^{n-k} \end{aligned}$$

where spectrum $(A_1) \subseteq$ imaginary axis, spectrum $(A_2) \subseteq \mathbb{C}^-$ and g_1 and g_2 are both C^2, $g_i(0,0) = 0$, $\frac{\partial g_i}{\partial y}(0,0) = \frac{\partial g_i}{\partial z}(0,0) = 0$ $i = 1,2$. (Thus $(0,0)$ is an equilibrium)

Then there exists $\delta > 0$ and C^1 function $h(y)$ defined for all y satisfying $\|y\| < \delta$, such that $z = h(y)$ is a center manifold, i.e. $h(0) = 0$ $\frac{\partial h}{\partial y}(0) = 0$ and $z = h(y)$ is an invariant manifold of $(*)$.

Proof: The proof of this existence theorem is an application of the contraction mapping – fixed point theorem on the space of maps h from $\mathbb{R}^k$ to $\mathbb{R}^{n-k}$ equipped with the sup norm

See H. Khalil, second edition appendix A.7. ☐

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The change of variables

$$\begin{pmatrix} y \\ z \end{pmatrix} \mapsto \begin{pmatrix} y \\ w \end{pmatrix} = \begin{pmatrix} y \\ z - h(y) \end{pmatrix}$$

turns the center manifold ~~so~~ constructed maps into the set $w = 0$ and invariance implies

$$\dot{w} \equiv 0$$

$$\Rightarrow \quad \dot{z} - \frac{\partial h}{\partial y}\dot{y} \equiv 0$$

$$\Rightarrow \quad A_2 h(y) + g_2(y, h(y)) - \left(\frac{\partial h}{\partial y}\right)\cdot(A_1 y + g_1(y, h(y))) \equiv 0$$

Thus the function h defining the center manifold satisfies the partial differential equation

$$A_2 h(y) + g_2(y, h(y)) = \frac{\partial h}{\partial y}\cdot(A_1 y + g_1(y, h(y)))$$

The dynamics in the y, w coordinates take the form (~ with a bit of adding / subtracting them

$$(**) \quad \boxed{\begin{aligned} \dot{y} &= A_1 y + g_1(y, h(y)) + N_1(y, w) \\ \dot{w} &= A_2 w + N_2(y, w) \end{aligned}}$$

where

$$N_1(y, w) \triangleq g_1(y, w + h(y)) - g_1(y, h(y))$$

$$N_2(y, w) \triangleq g_2(y, w + h(y)) - g_2(y, h(y)) - \frac{\partial h}{\partial y}(y)\, N_1(y, w)$$

mimic the arguments in Lecture 6 (part iii) pages 1-3 using the fundamental theorem of calculus leading upto $\|g(x)\|_2 < \gamma \|x\|_2$ where γ can be made arbitrarily small by picking $r > 0$ sufficiently small & $\|x\|_2 < r$. $\Uparrow$ But use partial derivative wrt w.

We have used here the partial differential equation satisfied by the center manifold.

It is easy to verify that N_1 and N_2 are both C^2 (since g_i are C^2), and $N_i(y,0) = 0$, $\frac{\partial N_i}{\partial w}(y,0) = 0$ $i = 1,2$.

Hence, in a domain $\left\|\binom{y}{w}\right\|_2 < \rho$, N_1 and N_2 satisfy

$$\|N_i(y,w)\|_2 \le k_i \|w\|_2 \qquad i = 1,2$$

where the positive constants k_i could be made arbitrarily small by choosing ρ small enough.

Restricted to the invariant manifold $w \equiv 0$, the dynamics $(**)$ takes the form

$$\dot{y} = A_1 y + g_1(y, h(y)) \qquad (\dagger)$$

which we call the <u>reduced system</u>. The equilibrium $(0,0)$ of the system $(*)$ projects to the equilibrium 0 of the reduced system $(\dagger)$. The main result of interest to us is that stability properties of the equilibrium of the unreduced system can be determined from those of the reduced system.

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Theorem (reduction and stability).

If the origin $y=0$ of the reduced system (*) is asymptotically stable (respectively unstable) then the origin of the full system (**) ~~(equivalently (***))~~ is asymptotically stable (respectively unstable).

Proof: First we note that if the origin of the reduced system is unstable, then any solution $y(t)$ of the reduced system, no matter how small $\|y(0)\|$ is, leaves the set $\{y : \|y\| < \varepsilon\}$ eventually. Hence the solution $(y(t), 0)$ of the full/unreduced system leaves the set $\{(y,w) : \|(y,w)\| < \varepsilon\}$ eventually. Thus $(0,0)$ is an unstable equilibrium of the unreduced system.

(aside: we have also shown that stability of the origin of the unreduced system $\Rightarrow$ stability of the origin of the reduced system).

Suppose that the origin of the reduced system is asymptotically stable. (The reduced system is autonomous. So one can specialize to the autonomous setting one of the converse Lyapunov theorems — in fact there is one best suited to this. See Hassan Khalil (Second Edition, Theorem 3.14) / (Third edition, Thm 4.16)

Then there is a C^1 function V such that

$V(y)$ is positive definite (i.e. $V(y) \geq \alpha_1(\|y\|_2)$ for a suitable class $\mathcal{K}$ function) and satisfies the following inequalities in a neighborhood of the origin,

$$\frac{\partial V}{\partial y}\left(A_1 y + g_1(y, h(y))\right) \leq -\alpha_3(\|y\|_2)$$

$$\left\|\frac{\partial V}{\partial y}\right\|_2 \leq \alpha_4(\|y\|_2) \leq k$$

where α_3 and α_4 are class $\mathcal{K}$ functions.

On the other hand, since A_2 is a Hurwitz matrix ($\mathrm{spectrum}(A_2) \subseteq \mathbb{C}^-$), the Lyapunov equation

$$A_2^T P + PA = -\mathbb{1}_{n-k}$$

has a unique positive definite solution $P = P^T$.

Consider $\quad W(y, w) = V(y) + \sqrt{w^T P w}$

as a Lyapunov function candidate for the full system (**). It is clearly a positive definite function. Along trajectories of (**)

$$\dot{W}(y, w) = \frac{\partial V}{\partial y}\left(A_1 y + g_1(y, h(y)) + N_1(y, w)\right)$$

$$+ \frac{1}{2\sqrt{w^T P w}}\left(w^T(A_2^T P + PA_2)w + 2w^T P N_2\right)$$

$$\leq -\alpha_3(\|y\|_2) + k k_1 \|w\|_2$$

$$- \frac{\|w\|_2}{\sqrt{\lambda_{max}(P)}} + k_2 \frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}} \|w\|_2$$

[At this step we used $\|N_i\|_2 \le k_i \|w\|_2$

$\|\frac{\partial V}{\partial y}\|_2 \le k$, $\lambda_{min}(P) \|w\|_2^2 \le w^T P w \le \lambda_{max}(P) \|w\|_2^2$]

Hence $\dot{W}(y, w) \le - \alpha_3 (\|y\|_2) - \frac{\|w\|_2}{4\sqrt{\lambda_{max}(P)}}$

$$- \left[\frac{1}{4\sqrt{\lambda_{max}(P)}} - k k_1 - k_2 \frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}} \right] \|w\|_2$$

Since k_i can be made arbitrarily small by restricting the domain around the origin (i.e by picking $\|w\|_2 < \rho$ and ρ small enough), we can choose ρ to insure that,

$$\frac{1}{4\sqrt{\lambda_{max}(P)}} - k k_1 - k_2 \frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}} > 0$$

Hence

$$\dot{W}(y,w) \le -\alpha_3(\|y\|_2) - \frac{1}{4\sqrt{\lambda_{max}(P)}}\|w\|_2$$

which implies asymptotic stability of the origin in the full system (**).

<u>Corollary 1</u> If the origin $y=0$ of the reduced system is stable, <u>and</u> there is a positive definite C^1 Lyapunov function $V(y)$ such that

$$\frac{\partial V}{\partial y}\left(A_1 y + g_1(y, h(y))\right) \le 0$$

in some neighborhood of $y=0$, then the origin of the full system is stable.

<u>Corollary 2</u> The origin of the full system (**) is asymptotically <u>iff</u> the origin of the reduced system is asymptotically stable.

The <u>reduction theorem</u> — a <u>User's Manual</u>

$\hat{h}$ satisfies the first order partial differential equation

$$N(\hat{h}(y)) \triangleq \frac{\partial h}{\partial y}\left(A_1 y + g_1(y, h(y))\right) - A_2 h(y) - g_2(y, h(y))$$

$$= 0.$$

with $h(0) = 0$ and $\frac{\partial h}{\partial y}(0) = 0$

Directly trying to compute the center manifold $h(\cdot)$ is <u>difficult</u>. If we could approximate $h(\cdot)$ and make statements based on an approximate reduced system then it is possible to use the reduction theorem as a practical tool

<u>Theorem</u> If a continuously differentiable function $\phi(y)$ with $\phi(0) = 0$ and $\frac{\partial \phi}{\partial y}(0) = 0$ can be found such that

$$N(\phi(y)) = O(\|y\|^p) \quad \text{for some } p > 1$$

then for sufficiently small $\|y\|$,

$$h(y) - \phi(y) = O(\|y\|^p)$$

and the reduced system can be represented as

$$\dot{y} = A_1 y + g_1(y, \phi(y)) + O(\|y\|^{p+1})$$

$\boxtimes$

How do we use this result?

The main idea behind the center-manifold reduction approach to stability assessment is to approximate the center manifold.

If an approximation at a certain order p does not work, ~~try~~ (i.e. is inconclusive on stability), then try a higher order approximation. The approximations to $h(y)$ are constructed by seeking $\phi(y)$ such that

$$\mathcal{N}(\phi(y)) = O(\|y\|^p),$$

where,

$$\phi(y) = h_2 y^{[2]} + h_3 y^{[3]} + \dots h_p y^{[p]}.$$

~~(recall $\phi(0) = 0$ & $\phi'(0) = 0$)~~

(The $y^{[k]}$ notation is explained in chapter 5 of Sastry's book – c.f. Carleman expansion/linearization –, and h_k are appropriate matrices satisfying the condition $\mathcal{N}(\phi(y)) = O(\|y\|^p)$). In the scalar case $y^{[k]} = y^k$. Substitute the ϕ into the formula for the reduced system

$$\dot{y} = A_1 y + g_1(y, \phi(y)) + O(\|y\|^{p+1})$$

for scalar y, $y^{[k]} = y^k$

and assess stability of the origin.

If this step is inconclusive try a higher p.

Example 1

$$\dot{x}_1 = x_2 \qquad\qquad a \neq 0$$

$$\dot{x}_2 = -x_2 + a x_1^2 + b x_1 x_2$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}. \quad \text{Thus} \quad k = 1.$$

Define
$$y = x_1 + x_2 \qquad \Longleftrightarrow \qquad x_1 = y + z$$
$$z = -x_2 \qquad\qquad\qquad x_2 = -z$$

$$\Rightarrow \quad \dot{y} = \underbrace{a(y+z)^2 - b(yz+z^2)}_{g_1(y,z)}$$

$$\dot{z} = -z \underbrace{- a(y+z)^2 + b(yz+z^2)}_{g_2(y,z)}$$

$$(\quad A_1 = 0 \; ; \; A_2 = -1)$$

$$\cancel{N(h(y)) = \frac{\partial h}{\partial y}\left(a(y+z)^2 - b(yz+z^2)\right)}$$

Center Manifold Defining Equation

$$N(h(y)) = \frac{\partial h}{\partial y}\left(a\,(y + h(y))^2 - b(y\,h(y) + h^2(y))\right)$$
$$+ h(y) + a(y + h(y))^2 - b(y\,h(y) + h^2(y))$$

$$h(0) = 0 \; ; \; h'(0) = 0.$$

Consider $p = 2$

$$\phi(y) = h_2 y^2$$

$$N(\phi(y)) = 2h_2 y\left(a(y + h_2 y^2)^2 - b(y h_2 y^2 + h_2^2 y^4)\right) + h_2 y^2 + a(y + h_2 y^2)^2 - b(y h_2 y^2 + h_2^2 y^4)$$

$$= O(|y|^2),$$

for <u>any</u> choice of h_2. One might try for simplicity $h_2 = 0$.

Then the reduced system is

$$\dot{y} = a y^2 + O(|y|^3)$$

For $a \neq 0$, the origin of the reduced system is unstable $\Rightarrow$ instability for the full system.

Here we used the property: For a scalar system $\dot{y} = a y^k + O(|y|^{k+1})$, k a positive integer,

$\rightarrow\cdot\leftarrow$ 0 is asymptotically stable if k odd and $a < 0$

$\leftarrow\cdot\rightarrow$ 0 is unstable if k odd and $a > 0$

$\rightarrow\cdot\rightarrow$ $\leftarrow\cdot\leftarrow$ or if k even and $a \neq 0$

Example 2

$$\dot{y} = yz \qquad g_1(y,z)$$

$$\dot{z} = -z + ay^2 \qquad g_2(y,z)$$

$A_1 = 0$

$A_2 = -1$

Center manifold equation

$$N(h(y)) = \frac{\partial h}{\partial y} \cdot (y\, h(y)) + h(y) - a y^2 = 0$$

$h(0) = h'(0) = 0.$

$$\phi(y) = h_2 y^2$$

$$N(\phi(y)) = 2h_2 y\,(y\, h_2 y^2) + h_2 y^2 - a y^2$$

$$= O(|y|^2) \qquad \text{for } \underline{\text{any}}\ h_2$$

Let us try $h_2 = 0.$

Reduced equation

$$\dot{y} = y\, h_2 y^2 + O(|y|^3)$$

$$= O(|y|^3)$$

which is <u>inconclusive</u> on stability

So try $\phi(y) = h_2 y^2 + h_3 y^3$

Then $$N(\phi(y)) = (2h_2 y + 3h_3 y^2)(y(h_2 y^2 + h_3 y^3)) + (h_2 y^2 + h_3 y^3) - a y^2$$

$$= O(|y|^3) \qquad \text{only if we choose } h_2 = a$$

In that case the reduced system is

$$\dot{y} = y(ay^2 + h_3 y^3) + O(|y|^4)$$

$$= ay^3 + O(|y|^4)$$

for which 0 is asymptotically stable provided $a<0$

0 is unstable if $a>0$

Consequently the origin is asymptotically stable for the full system if $a<0$ & unstable for the full system if $a>0$

Example 3

This example illustrates the case where the center manifold dimension is >1.

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} -y_1^3 \\ -y_2^3 + z^2 \end{pmatrix}$$

$$\dot{z} = -z + (y_1^3 - 3y_1^5 + 3y_1^2 y_2)$$

Try $\phi(y) = 0 \Rightarrow N(\phi(y)) = O(\|y\|_2^3)$

and
$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} -y_1^3 + y_2 \\ -y_2^3 - y_1 \end{pmatrix} + O(\|y\|_2^4)$$

Let $V(y) = \frac{1}{2}(y_1^2 + y_2^2)$.

$$\dot{V} = -y_1^4 - y_2^4 + y^T O(\|y\|_2^4)$$

$$\leq -\|y\|_2^4 + k \cdot \|y\|_2^5$$

in some neighborhood of the origin where $k > 0$. Hence

$$\dot{V} \leq \frac{1}{2}\|y\|_2^4 \quad \text{for } \|y\|_2 < \frac{1}{2k}$$

$\Rightarrow$ 0 is asymptotically stable ▨