The transfer function of the system
\[ \tilde{x} = (A - BK_{\text{min}} C)x + Bu \]
\[ y = Cx \]
is the same as the transfer function of the closed loop system in the adjoining block diagram.

![Block Diagram](image)

where \( G(s) = \frac{C}{(sI - A)^{-1}B} \) as before.

Observe that, in terms of Laplace transforms of input and output,
\[ Y(s) = G(s) E(s) \]
\[ E(s) = U(s) - K_{\text{min}} Y(s) \]

\[ Y(s) = G(s) U(s) - G(s) K_{\text{min}} Y(s) \]
\[ (1 + G(s) K_{\text{min}}) Y(s) = G(s) U(s) \]
\[ Y(s) = (1 + G(s) K_{\text{min}})^{-1} G(s) U(s) \]
\[ \tilde{G}(s) = (1 + G(s) K_{\text{min}})^{-1} G(s) \].

On the other hand,
\[ E(s) = U(s) - K_{\text{min}} Y(s) \]
\[ = U(s) - K_{\text{min}} C(s) E(s) \]
\[ \begin{align*}
\Rightarrow \quad \tilde{E}(s) &= (1 + K_{\text{min}} G(s))^{-1} \tilde{U}(s) \\
\Rightarrow \quad E(s) &= (1 + K_{\text{min}} G(s))^{-1} U(s) \\
\Rightarrow \quad Y(s) &= G(s) E(s) \\
&= G(s) (1 + K_{\text{min}} G(s))^{-1} \tilde{T}(s)
\end{align*} \]

So we have two formulas:

\[ \tilde{E}(s) = (1 + A(s) K_{\text{min}})^{-1} G(s) \]

\[ = G(s) (1 + K_{\text{min}} G(s))^{-1} \]

(and of course they are equivalent as a little algebra shows).

Now \( A - BK_{\text{min}}C \) is Hurwitz if all the poles of \( \tilde{E}(s) = G(s) (1 + K_{\text{min}} G(s))^{-1} \)

are in \( \mathbb{C}^- \).

Applying the sector condition to \( \tilde{Y} \):

\[ \tilde{Y}(t, y) \top (\tilde{Y}(t, y) - K_{\text{min}} y) \leq 0 \]

\[ \Leftrightarrow (Y(t, y) - K_{\text{min}} y) \top (Y(t, y) - (K_{\text{min}} + K)y) \leq 0 \]

\[ \Leftrightarrow (Y(t, y) - K_{\text{min}} y) \top (Y(t, y) - K_{\text{max}} y) \leq 0 \]

for \( K_{\text{max}} = K_{\text{min}} + K \).
The relevant positive real transfer function is

\[ Z(s) = \frac{1}{1 + K \tilde{G}_0(s)} \]

\[ = \frac{1}{1 + K \alpha (1 + K_{\text{min}} G)^{-1}} \]

\[ = \frac{(1 + K_{\text{min}} G)(1 + K_{\text{min}} G)^{-1}}{1 + K \alpha (1 + K_{\text{min}} G)^{-1}} \]

\[ = (1 + K_{\text{min}} + K) G \left( \frac{1}{1 + K \alpha} \right) \]

\[ = (1 + K_{\text{max}} G)(1 + K_{\text{min}} G)^{-1} \]

Now we are ready to state the multivariable circle criterion without the Hurwitz assumption.

**Theorem**

Let \([A, B, C]\) be a controllable and observable circle criterion triple. Suppose \(y\) satisfies the sector condition

\[ (y^T v, y) \geq K_{\text{min}} \]

\[ + 0 \leq y \leq \mathbb{R}^m \text{ and } K = K_{\text{max}} - K_{\text{min}} = K^T > 0 \text{ given} \]

Then the \( \text{closed-loop system} \) is absolutely stable provided

(a) \( \tilde{G}(s) = (s I + K_{\text{min}} G(s))^{-1} \)

\( \text{"Hermite" (analytic in } \mathbb{R} \text{ if } s + \text{Re}(s) > 0)\)

(b) \( \tilde{Z}(s) = \left( \frac{1}{1 + K_{\text{max}} G}(1 + K_{\text{min}} G)^{-1} \right) \)

\( \text{in strict positive real.} \)
Proof: From the remarks preceding the above statement, it is clear that all one has to do is to appeal to the equivalence of closed loop systems with and without the loop transformation arising from the feedback $A + B$ from $C$ and appeal to the Hurwitz case already proved.

Where does the name "circle criterion" come from? This is an interesting story going back to the work of Harry Nyquist, the AT&T Mathematician who investigated graphical methods for feedback amplifier stability in long-distance (transatlantic) telephony. This is a direct application of the principle of the argument in complex variable theory.

First specialize to the single input, single output case.

Let $T(s) = \{ u + jw = G(jw) \mid \omega \in \mathbb{R}_+ \}$

be the Nyquist locus of $G$. Let $G$ be proper and

Theorem (Nyquist) Let $T(s)$ be bounded (i.e. no poles on the imaginary axis). We will say that $T(s)$ encircles
a point \( z_0 + j\omega \), \( p \) times, if \( z_0 + j\omega \) is not on \( T_0 \) and \( 2\pi p = \text{net increase} \) in the argument of \( G(z) = (z_0 + j\omega) \) as \( z \) increases from \(-\infty\) to \( +\infty\). 

Clockwise encirclement or direction of increasing argument

Counter-clockwise encirclement or direction of decreasing argument

Suppose \( T_0 \) is bounded. If \( G \) has \( r \) poles in the half plane \( C^+ \), then \( \frac{G}{1 + KG} \) has \( p + r \) poles in the half plane \( C^+ \) if the point \( -\frac{1}{K} + j\omega \) is not on \( T_0 \) and \( T_0 \) encircles it \( p \) times in the clockwise sense.

(proof: see Frankel et al. \( \rightarrow \) reference list for this class)

Corollary: If \( T_0 \) is bounded and \( -\frac{1}{K} + j\omega \) is not on \( T_0 \) and \( G \) has \( r \) poles in \( C^+ \) then the feedback \( u = -K\hat{y} \) stabilizes the closed loop system if \( T_0 \) encircles \( \frac{1}{K} + j\omega \) \( p \) times in the counter-clockwise direction.
Lemma: Let $g(s)$ be a scalar transfer function. Let $g(s)$ be proper (i.e., $g(s) = \frac{y(s)}{p(s)} + d$) where $\deg(y) < \deg(p)$, $p$ monic, and $d$ a constant. The poles of $g(s)$ all lie in $\mathbb{C}^-$. Then $g(s)$ is positive real iff $\Re(g(j\omega)) > 0$ for $\omega \in \mathbb{R}$.

Proof: < See H. Khalil > page 404.

Theorem: Let $g(s)$ be a scalar transfer function $c(sI-A)^{-1}$, $[A, b, c]$ controllable and observable. Let $y(t, y)$ satisfy the sector condition:

$$\alpha y^2 \leq y y(t, y) \leq \beta y^2$$

Then absolute stability of the closed loop system

$$\dot{x} = Ax + bu$$

$$y = cx$$

$$u = -y(t, y)$$

holds provided one of the following conditions apply:

(i) If $0 < \alpha < \beta$, the Nyquist locus does not enter the disk $D(0, \beta)$ and encircles it $n$ times in the counterclockwise direction where

$$n = \# \text{ poles of } g(s) \text{ in } \mathbb{C}^+$$
(ii) If \( 0 < \alpha < \beta \), \( \mathcal{G}(s) \) is "Hurwitz" and the Nyquist plot \( \mathcal{N}(s) \) lies to the right of the line \( \Re(s) = -\frac{1}{\beta} \).

(iii) If \( \alpha < 0 < \beta \), \( \mathcal{G}(s) \) is "Hurwitz" and the Nyquist plot \( \mathcal{N}(s) \) lies in the interior of the disk \( \mathcal{D}(0, \beta) \).

Proof: Specializing the multivariable circle criterion to this case, we seek conditions to ensure that:

(a) \( \frac{\mathcal{G}(s)}{1 + \alpha \mathcal{G}(s)} \) is Hurwitz and

(b) \( \frac{1 + \beta \mathcal{G}(s)}{1 + \alpha \mathcal{G}(s)} \) is strict positive real.

For (b), it is equivalent to check

\[
\Re \left( \frac{1 + \beta \mathcal{G}(j\omega)}{1 + \alpha \mathcal{G}(j\omega)} \right) > 0 \quad \forall \omega \in \mathbb{R}
\]

In case (i): \( 0 < \alpha < \beta \), this is equivalent to checking

\[
\Re \left( \frac{1 + \beta \mathcal{G}(j\omega)}{1 + \alpha \mathcal{G}(j\omega)} \right) > 0 \quad \forall \omega \in \mathbb{R}
\]

Consider the figure:

\[
\theta_1 = \arg \left( \frac{1}{\beta} + \mathcal{G}(j\omega) \right)
\]

\[
\theta_2 = \arg \left( \frac{1}{\alpha} + \mathcal{G}(j\omega) \right)
\]

\[
\theta = \theta_1 - \theta_2
\]
$$\text{Re} \left( \frac{1}{\beta} + g(i\omega) \right) = \text{Re} \left( \frac{1}{\beta} \right) + g(i\omega)$$

where \( \gamma > 0 \) or \( \gamma > 0 \) \( \Rightarrow \) \( \beta = \theta - \frac{i}{\mu} \)

By elementary geometry, \( \gamma \) has to lie outside \( D(\gamma, \beta) \) the disc with diameter joining \((-1, 0)\) and \((-1, 0)\). For the encirclement condition use the corollary to Nyquist.

In case (ii) the condition for strict positive reality is:
$$\text{Re} \left( \frac{1}{\beta} + g(i\omega) \right) > 0$$

$$\Leftrightarrow \cos (\theta) > 0$$

$$\Leftrightarrow \frac{\pi}{2} < \theta < \frac{3\pi}{2}$$

$$\Leftrightarrow \frac{T_1}{2} \text{ lies to the right of the vertical line }$$

\( i.e. \text{Re}(s) = -\frac{1}{\mu} \)

In case (iii) same arguments as in (i) above but we seek \( \gamma > \frac{\pi}{2} \) \( \Leftrightarrow \gamma \) lies in the interior of the disc \( D(\gamma, \beta) \).

\( \Rightarrow \) because \( \gamma \) and \( \beta \) have opposite sign the strict positive reality condition in
$$\text{Re} \left( \frac{1}{\beta} + g(i\omega) \right) < 0$$
The Paper criterion for absolute stability

is based on

(i) restrictions on \( y \)

(ii) use of non-quadratic Lyapunov function

\[
\begin{align*}
(i) \quad y &= v(y) = (v_1(y), v_2(y), \ldots, v_m(y))^T \\
&\quad v^T(y) P (y - Ky) \leq 0 \\
&\quad K = \text{diag}(\beta, \ldots, \beta), \quad \beta > 0 \quad + \tag{1} \\
(ii) \quad V(x) &= x^T P x + 2\eta \int_0^\infty \sum_{i=1}^m v_i(x_0) P_i \, dx_0 \\
&\quad \leq x^T P x + 2\eta \int_0^\infty y^T K \, dy \\
By \text{sector condition, } y(x_0) \neq 0 \quad + \\
&\quad \Rightarrow \quad y(x_0) \neq 0 \quad (\text{as long as } P = P^T > 0) \\
Along\, \text{trajectories of the (usual) closed loop system} \\
\dot{v} &= x^T P x + x^T P x + 2\eta y^T K y \\
&= (Ax - By)^T P x + x^T P (A - By) \\
&\quad + 2\eta y^T K C (Ax - By) \\
&= x^T (AT + PA) x - 2x^T PB y \\
&\quad + 2\eta y^T K C (Ax - By) \quad + 2\eta y^T K C (Ax - By) \\
Since \quad -2y^T (y - Ky) > 0 \quad we \quad get \\
\dot{v} &\leq x^T (AT + PA) x - 2x^T PB y + 2\eta y^T K C (Ax - By) \\
&\quad - 2y^T (y - Ky) \\
&= x^T (AT + PA) x - 2x^T (PB - \eta AT C^T K - CK) y \\
&\quad - y^T (2A + \eta KCB + \eta B^T CK) y \
\end{align*}
\]
Choose \( \eta \) small enough s.t.
\[
2\eta + \eta KCB + \eta B^Tc^T K > 0
\]
\[
\eta \text{ we can find } W \text{ s.t.}
\]
\[
w^T w = 2\eta + \eta KCB + \eta B^Tc^T K
\]
\[
= (A + \eta KCB)^T + (\_\_)^T
\]
Suppose \( \eta > 0 \) and \( \exists \ L \) and \( \epsilon > 0 \) s.t.
\[
AP + PA = -LT - \epsilon \text{P}
\]
\[
Pb = c^T K + \eta A^T c^T K - L^T W
\]
\[
= (C + \eta CA)^T K - L^T W
\]
Then
\[
\dot{v}(x) \leq -\epsilon x^T P x - x^T L^T L x + 2 x^T L^T W y - y^T W^T W y
\]
\[
= -\epsilon x^T P x - (Lx - Wy)^T (Lx - Wy)
\]
\[
\leq -\epsilon x^T P x
\]
\[
< 0 \quad x \neq 0
\]
Thus we get absolute stability. The question of \( P, L, c, W \) is settled by the KYP lemma.
\[
Z(s) = (A + \eta KCB) + (KC + \eta KCA)(sI - A)^{-1} B
\]
\[
= A + \eta KC (sI - A + A)(sI - A)^{-1} B
\]
\[
+ KC (sI - A)^{-1} B
\]
\[
= 11 + \eta s KC (sI - A)^{-1} B + KC (sI - A)^{-1} B
\]
\[
= 11 + (\eta s + 1) K G_1(s)
\]
Suppose $\gamma$ is chosen that $-1$ is not an eigenvalue of $A$. Then $(A, K(c + \gamma CA))$ is observable if $(A, c)$ is observable.

Then by KYP, $P, L, E$ exist if

$$M(s) = \frac{1}{s^2 + (\gamma + j) K G(s)}$$

is strictly positive real. We have proved

**Theorem (Multivariable Popov Criterion)**

\[
\dot{x} = Ax + Bu \\
y = cx \\
u = -y^T(y) \\
k = (\beta_1, \ldots, \beta_m)^T
\]

Then $(*)$ is absolutely stable if $\exists Y > 0$ s.t. $-1 \in \text{spectum}(A)$ and

$$Z(s) = I_m + (s + j\beta_1 Y(s)) K G(s)$$

is strictly positive real.
Choose \( \eta \) small enough so that
\[ 2 \eta + \eta KCB + \eta B^T C K > 0 \]
\( \Rightarrow \) we can find \( W \) s.t.
\[ W^T W = 2 \eta + \eta KCB + \eta B^T C K = (\eta + \eta KCB)^T \]

Suppose \( P = P^T > 0 \) and \( J \) L and \( \eta > 0 \) s.t.
\[ AT \eta P + PA = -L^T L - \eta P \]
\[ PB = C^T K + \eta A^T C K - L^T W = (C + \eta CA)^T K - L^T W \]

Then,
\[ \psi(x) = -\eta x^T P x - x^T L^T L x + 2x^T L^T W y - y^T W^T W y \]
\[ = -\eta x^T P x - (Lx - Wy)^T (Lx - Wy) \]
\[ \leq -\eta x^T P x \]
\[ < 0 \quad x \neq 0 \]

Thus we get absolute stability. The question of

\( P, L, \eta, W \) is settled by the KYP Lemma.

\[ Z(s) = (\eta + \eta KCB)^T + (KC + \eta KCA)(sI - A)^{-1} B \]
\[ = (\eta + \eta KC)(sI - A + A)(sI - A)^{-1} B + KC(sI - A)^{-1} B \]
\[ = (\eta + \eta KC)(sI - A)^{-1} B + KC(sI - A)^{-1} B \]
\[ = \eta(\eta + sI)K \cdot G(s) \]
Suppose \( \gamma \) is chosen so that \(-1\) is not an eigenvalue of \( A \). Then \((A, K(C + \gamma CA'))\) is observable if \((A, C)\) is observable.

Then by KYP, \(P, L, E\) exist if

\[
\Xi(s) = \frac{1}{s} + (1 + s)K G(s) \in \text{strict positive real. We have proved.}
\]

**Theorem (Multivariable Popov criterion)**

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx \\
u &= -y(y)K = \text{diag}(\beta_1, \ldots, \beta_m) \quad \beta_i > 0, i = 1, \ldots, m \\
0 &\leq y, y(y) \leq \beta_i y_i^2 \quad \text{(sector condition)}
\end{align*}
\]

Then \((x)\) is absolutely stable if \( \exists \gamma > 0 \) s.t. \(-\frac{1}{\gamma} \in \text{specturm (A)}\) and

\[
\Xi(s) = \frac{1}{s} + (1 + s)K G(s) \in \text{strict positive real.}
\]
Remark (a) With $\eta = 0$, this reduces to a special case of the circle criterion.

(b) With $\eta > 0$, we get absolute stability under weaker conditions (but for a restricted class of non-linear maps $\eta$).

(c) For $m = 1$ (zero case), we have a graphical test.

Choose $\eta$ s.t. $Z(\infty) = \lim_{\xi \to \infty} Z(\xi) = \omega > 0$.

Then $Z(\xi)$ is strict positive real iff

$$\text{Re} \left[ \frac{1}{1 + \eta \omega} \text{Re}(g(i\omega)) - \eta \omega \text{Im}(g(i\omega)) \right] > 0 \quad \forall \omega \in \mathbb{R}$$

Note $k > 0$ implies

$$\frac{1}{k} + \text{Re}(g(i\omega)) - \omega \text{Im}(g(i\omega)) > 0 \quad \forall \omega \in \mathbb{R}$$

$\iff$ Popov locus plot lies to the right of the line that intercepts the point $-k + j0$ with slope $\eta$.

Here, Popov locus $P_{\eta} = \{ u + jv : u = \text{Re}(g(i\omega)) \}$

$$v = \omega \text{Im}(g(i\omega)) \quad \forall u \in \mathbb{R}$$

$P_{\eta}$ is the locus of all points $u + jv$ that satisfy the condition $P_{\eta}$.