Lecture 5 (part 1)

The principle of mechanics as developed in the work of Euler, Laplace, and Dirichlet lead to the tools for understanding the stability properties of solutions of nonlinear systems. The modern approach to the problem of the stability of dynamical systems is based on the classical works of A.M. Lyapunov (Problème général de la stabilité des mouvements, Ann. Éc. Éc., 1892, fascimile published in Princeton, 1947, 1949). The key ingredient of Lyapunov's approach is the introduction of a function of the state variables, called the Lyapunov function, which decreases along trajectories of the system. An equilibrium point is a point where the Lyapunov function is constant, and the system evolves in the vicinity of the equilibrium point according to the equations of motion.

1. Define $V(x)$ as a Lyapunov function. Let $y = x - x^*$ be the distance from the equilibrium point.
2. Compute $V(x(t)) = V(x)$ and $y(t)$ as functions of time.

This is a standard tool in most textbooks on the stability of dynamical systems. For a system (1), the equilibrium point $x^*(t)$ is a stationary point of the system, and any other trajectory $y(t)$ maps onto a trajectory of the system (2). Thus, the Lyapunov function $V(x)$ decreases along trajectories of the system (2), indicating the stability of the equilibrium point (1).
Equilibrium point \( x^* \).

Definition (Lagrange multiplier):

Let \( f \) be a differentiable function for all \( x \in E^m \). Define \( L(x, \lambda) = f(x) - \lambda g(x) \). The gradient of \( L \) is zero at a point \( x^* \) if and only if the solution to the system of equations

\[
\nabla L(x^*, \lambda) = 0,
\]

satisfies \( g(x^*) = 0 \).

(1) The equation \( x^* \) is said to be stable in the sense of Lagrange.
Theorem (Lyapunov)

Let $D$ be a domain containing $0 \in \mathbb{R}^n$ and $f$ be a $C^1$ function.

Let $x_0$ be an equilibrium point of the system $\dot{x} = f(x)$.

Let $V(x)$ be a $C^2$ function such that $V(x_0) = 0$, $V(x) > 0$ in $D - \{x_0\}$, $V(x) \leq 0$ in $D$, and $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\|x - x_0\| < \delta \Rightarrow V(x) < V(x_0)

Then $x_0$ is stable. Moreover, if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\|x - x_0\| < \delta \Rightarrow \|\dot{V}(x)\| < \epsilon \|x - x_0\|$, then $x_0$ is asymptotically stable.

Proof: The adjoint of $x$ is connected with the proof.

Given $\epsilon > 0$, choose $r \in (0, \epsilon)$. Then $x_0$ is asymptotically stable.
Let $a < b < c$. 

Then $a < b < c$.

Proof: Suppose not. Let $a > b$.

Then $a$, $b$, $c$.

Take $p$ such that $0 < p < a$, and let

$p$.
There are some abbreviations in the call-ordary of $\mathbb{R}$.
Suppose \( H(x) \) is a function of \( x \), then

\[
\frac{d}{dx}(xH(x)) = xH(x) - H(x) + (xH(x))'
\]

for all \( x \in \mathbb{C} \). Now, suppose \( H(x) = (\theta_x + \theta)^2 \) for \( \theta \neq 0 \).

If we further assume that

\[
\frac{d}{dx} \theta = 0
\]

and

\[
\theta_e = \theta
\]

the state space can be factored into position

\[
\theta_e = \theta
\]

Note that \( \dot{\mu}(x) = \frac{x}{\mu} \). Suppose

\[
\mu \neq 0 \quad \text{and} \quad \dot{x}
\]

is the Lagrangian function. \( L \) is a Lagrangian for \( x \).

However, the above result is not a contradiction of ordinary differential equations, as it only applies in the case of fixed points of the energy method. In the theory of Green's functions, this is a fundamental fact.
The theorem - Conclusion
The result isn't derived in Lemma or the

Theorem 2.12

Proof of Theorem 2.12

Proof:

To prove that \( H(x) \) is a graph of a function, consider the following:

Given \( (x, y) \in H \),

- \( x \) is finite.
- \( y = H(x) \) is unique.

Therefore, \( H \) is a function.

For any \( x \),

- If \( x = 0 \), then \( y = 0 \).
- If \( x \neq 0 \), then \( y = \frac{1}{x} \).

Consequently, \( H(x) \) is a function.
\[ \frac{\partial}{\partial \theta} \theta = 0 \]

Arguments. The argument for the computer science subject by Laplace. 2. Neither one can nor all

Small angle of \( \theta \), use the principle of symmetrical

Only two equations in C, thus 2 \( \theta \) can be.

So, we can set \( \theta \), such that \( (\theta, 0) \).

By definition:

\[ \theta > 0 \]

\( \alpha \) \( (\theta, \alpha) \) \( \alpha \) \( \beta \) \( \beta \)

Alloy. Theorems of \( (\theta) \)

\[ \text{If} \ (1, 1) = \frac{1}{2} \text{ and } (1, y) = 0 \]

\[ \text{Define: the region } \]

\[ P = R - R (y) \]

\[ p = \frac{\partial R}{\partial x} - p \]

\[ q = -w \]

No modification:

Conclusion from the facts or its description.

In classical physics, modern dynamics, and

Mechanical System. A very common of systems.

Hamilton's equations of the form (1)

\( \text{P.S.K} \)

09.30.00
\[ dP = \frac{d}{dt} \]

\[ V(t) = \frac{1}{\theta} (1 - \cos(\theta)) \]

\[ R(t) = b = b > 0 \]

Example (Finding Laplace Transform) ~ decomposed function.

(0, 26)
\[ y = \frac{1}{2} p^2 + 8 q - \lambda x \left( \frac{q}{2} \right) \]

Then, \( y \) is a Lyapunov function.

\[ \frac{\partial}{\partial y} \left( \frac{q}{2} \right) = \frac{1}{2} p^2 + 8 q - \lambda x \left( \frac{q}{2} \right) \]

Since \( \frac{\partial}{\partial y} \left( \frac{q}{2} \right) > 0 \), we can conclude that \( y \) is positive definite.

We define \( \epsilon > 0 \) such that

\[ \frac{\partial}{\partial y} \left( \frac{q}{2} \right) = \frac{1}{2} p^2 + 8 q - \lambda x \left( \frac{q}{2} \right) > \epsilon \]

Since \( \epsilon > 0 \), we have

\[ \frac{\partial}{\partial y} \left( \frac{q}{2} \right) > 0 \]

Thus, the system is asymptotically stable.
Example (Gradient Dynamics)

\[ \mathbb{R}^n \to \mathbb{R} \]

for all \( x \in \mathbb{R}^n \rightarrow f(x) \in \mathbb{R} \).

For all \( x \in \mathbb{R}^n \rightarrow f(x) \leq 0 \).

The unique global solution of LaSalle's principle of dynamical systems is based on the feedback of the 20th-century work of G.D. Birkhoff, American mathematician of the complex theory of dynamical systems.

Theorem (Birkhoff) If a trajectory \( x(t) \to 0 \),

\[ \lim_{t \to \infty} x(t) = 0 \]

Moreover, the system is a hyperbolic, compact, invariant set.

\[ \mathbb{R}^n \to \mathbb{R} \]

and bounded.
We omit the proof of Birkhoff's Theorem (see for instance H. Khalil).

Theorem (LaSalle) Let $\Omega$ be a compact (closed and bounded) set with $E \subset \Omega \rightarrow \mathbb{R}$ a $C^1$ function.

Let $E$ be the set of all points in $\Omega$ where $v(\alpha) = 0$.

Let $l$ be the solution starting in $\Omega$ tends to 1 as $t \to \infty$.

Since $v(t)$ is a solution of $t$, $v(z(t))$ is a monotone decreasing function of $t$. Since $v(z(t))$ is continuous on $\Omega$. Therefore $v(\alpha)$ is closed. For any $r \in \Omega$

By continuity of $v$, $v(\alpha) = \lim v(z(t)) = \alpha$.
Proof

\[ W = \left\{ (x',y') \mid (x,y) \in W \right\} \]

Exercise

For \( c > 0 \) such that \( H(x',y') = c \) for all \( c \), consider

\[ E = \left\{ (x',y') \mid (x,y) \in H(x',y') \right\} \]

every \( f \), \( f \) is continuous.

\[ \frac{\partial}{\partial x} H(x',y') = 0 \text{, } c > 0 \]

such that, \( H(x',y') \) and \( c \).

In some problems one can pick \( \varphi \) and \( \zeta \).

\[ \text{Remark} \]

In many problems it is much easier

\[ f(x) \to -\infty \text{ as } x \to -\infty \]

\[ f(x) \to +\infty \text{ as } x \to +\infty \]

Since \( f(x) \) is bounded, \( \exists c \text{ such that } c \leq f(x) \leq c + L \).

Since \( W \) is the largest compact set, \( E \), we get

\[ \text{Invariance (Lyapunov)} \]

\[ \exists x = 0 \text{ on } L + \]

\[ \text{Hence } L + c \]

\[ \text{Step 0} \]
\[
E = \frac{1}{2} (\mathbf{v}(t) \cdot \mathbf{v}(t)) \in \mathbb{R}^2 \quad \beta = 0
\]

\[
\frac{d}{dt} (\mathbf{v}(t)) = 0 \quad (\text{from the dynamics})
\]

\[
\Rightarrow \quad \beta(t) = 0 \quad (\text{from the dynamics})
\]

\[
\Rightarrow \quad \frac{\partial}{\partial \mathbf{v}(t)} (\mathbf{v}(t) \cdot \mathbf{v}(t)) = 0 \quad (\text{from the above})
\]

Thus we have shown that \((q, \beta) \in M\)

implies \(\beta = 0\) and \(q\) is a critical point of \(V\). Choosing a small \(\epsilon\), we can

establish that the projection of \(\{x(0) = \mathbf{0}\}\) is \(V\). Then we have shown

that any trajectory starting in \(O\)

\[
\Rightarrow \quad (\mathbf{v}(0), \mathbf{0})
\]