ENEE 661 Nonlinear Control Systems

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Lecture 5 (part i)

Lyapunov = Liapounoff = Liapunov

The principles of mechanics as developed in the works of Laplace, Lagrange, and Dirichlet lead to tools for understanding the stability properties of solutions of nonlinear systems. The modern period in this direction begins with the classic memoir of A.M. Lyapunov (Problème générale de la stabilité de mouvement, Ann. ~~Sci~~ Fac. Sci. Toulouse, 9: 203-474 (1907) — translation of a paper in Russian published in Comm. Soc. Math. Kharkov 1892, facsimile reproduction as Annals of Mathematics Study No. 17 (Princeton Univ. Press, 1947)). The key ingredients are: (a) a definition of stability and, (b) an "energy method" to assess stability. We introduce these in the setting of autonomous differential equations.

An equilibrium point x_e of a system

(1) $$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

satisfies $f(x_e) = 0$. Let $y = x - x_e$ and define $g(y) = f(y + x_e)$. Then the equilibrium point $y = 0$ of the system

(2) $$\dot{y} = g(y)$$

corresponds to the equilibrium point x_e of the system (1), and any other trajectory $x(t)$ of (1) maps onto a trajectory $y(t) = x(t) - x_e$ of the system (2). Thus the shift of origin in $\mathbb{R}^n$ allows one to refer to an equilibrium at 0. This is a standard device in most treatments — we don't use it below.

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<u>Definition</u> (Lyapunov stability)

(i) The equilibrium x_e of (1) is said to be <u>stable in the sense of Lyapunov</u>, if given $\varepsilon > 0$, there exists $\delta > 0$ such that

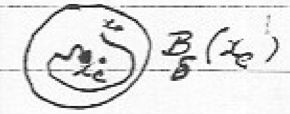
$$x_0 \in B_\delta(x_e) \quad \text{implies that}$$

the solution starting at x_0, denoted $x(t)$, is <u>trapped</u> in $B_\varepsilon(x_e)$:

$$x(t) \in B_\varepsilon(x_e) \quad \forall \quad t \geq 0.$$

Here $B_r(z)$ stands for the <u>open</u> ball of radius r centered at z — $B_r(z) := \{x : \|x - z\| < r\}$, with respect to a choice of norm $\|\cdot\|$ in $\mathbb{R}^n$.

<u>Figure for (i)</u>



<u>Figure for (iii)</u>

(ii) x_e is unstable if not stable.

(iii) x_e is <u>asymptotically stable</u> if it is stable as in (i) <u>and</u> $\delta > 0$ can be chosen such that $\lim_{t \to \infty} x(t) = x_e$, for all initial condition $x_0 \in B_\delta(x_e)$ (attractivity of x_e) ☒

The basic theorem of the subject, due to Lyapunov gives a sufficient condition for stability (or asymptotic stability) of an equilibrium point x_e.

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<u>Theorem</u> (Lyapunov)

Let x_e be an equilibrium point of the system (satisfying local Lipschitz condition)

$$\dot{x} = f(x).$$

Let D be a domain (= open connected region) of $\mathbb{R}^n$ containing x_e. Suppose $V: D \to \mathbb{R}$ is a C^1 function and

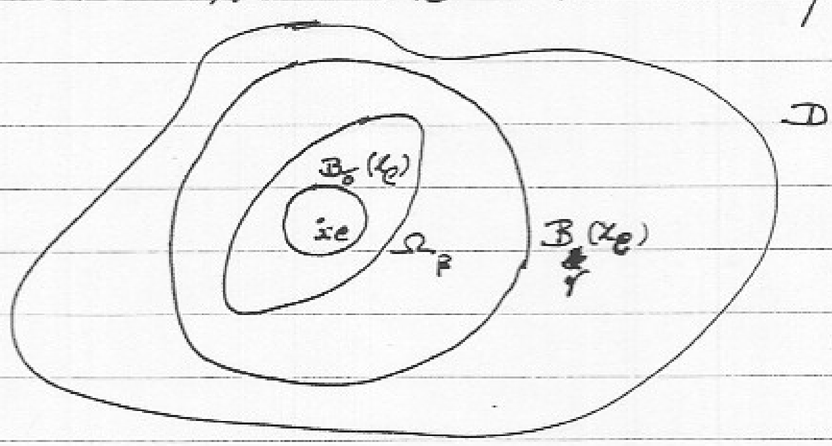
$$V(x_e) = 0$$
$$V(x) > 0 \quad \text{in} \quad D - \{x_e\}$$
$$\dot{V}(x) \leq 0 \quad \text{in } D.$$

Then x_e is stable. Moreover, if

$$\dot{V}(x) < 0 \quad \text{in} \quad D - \{x_e\}$$

then x_e is asymptotically stable.

<u>Proof</u> The adjoining figure should be referred to in connection with the nested sets used in the proof.



Given $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ such that closed ball $\overline{B}_r(x_e) = \{x : \|x - x_e\| \leq r\} \subset D$.

Let $\alpha = \min_{\|x - x_e\| = r} V(x)$. Then $\alpha > 0$ by hypothesis.

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Take β such that $0 < \beta < \alpha$, and let

$$\Omega_\beta = \{ x \in \bar{B}_r(x_e) : V(x) \le \beta \}.$$

Then $\Omega_\beta \subset B_r(x_e)$.

proof: suppose not. Let $p \in \Omega_\beta$ be such that $\| p - x_e \| = r$. Then $V(p) \geq \alpha > \beta$, a contradiction.

Let $x(t)$ be a solution with $x(0) \in \Omega_\beta$. Since $\dot{V}(x(t)) \le 0$, $V(x(t)) \le V(x(0))$ and hence $x(t) \in \Omega_\beta$, $\forall\, t \geq 0$. Since Ω_β is a closed and bounded set, we conclude (see Theorem, Lecture Notes 4, page 16) that we have a unique solution $\subset \Omega_\beta$, a positively invariant set, for all time.

V is continuous at x_e and $V(x_e) = 0$, implies there is a $\delta > 0$ such that

$$\| x - x_e \| \le \delta \implies V(x) < \beta .$$

Thus,

$$B_\delta(x_e) \subset \Omega_\beta \subset B_r(x_e),$$

and

$$x(0) \in B_\delta(x_e) \implies x(0) \in \Omega_\beta$$
$$\implies x(t) \in \Omega_\beta \quad \forall\, t \geq 0$$
$$\implies x(t) \in B_r(x_e) \quad \forall\, t \geq 0$$

(since $\varepsilon \geq r$)
$$\implies x(t) \in B_\varepsilon(x_e) \quad \forall\, t \geq 0$$

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Under the extra assumption $\dot V(x)<0$ in $D-\{x_e\}$, one can show that the <u>monotone decreasing</u> function $V(x(t))$ has a limit $c=0$. The existence of a limit $c \geq 0$ is assured by the lower bound on V. <u>Suppose</u> $c>0$. By continuity of V there is a $d>0$ such that

$$B_d(x_e) \subset \Omega_c .$$

[Here we are appealing to the 'well-ordering' of $\mathbb{R}$]

Since $\lim_{t\to\infty} V(x(t)) = c > 0$ by hypothesis,

$$x(t) \text{ never enters } B_d(x_e).$$

Let
$$-\gamma = \max_{d \leq \|x-x_e\| < r} \dot V(x)$$

By hypothesis, $\gamma > 0$. By the fundamental theorem of integral calculus,

$$V(x(t)) = V(x(0)) + \int_0^t \dot V(x(\tau))\,d\tau$$

$$\leq V(x(0)) - \gamma t$$

Since, the right hand side eventually becomes negative, we get a contradiction from assuming $c>0$. So $c=0$. Hence

$$\lim_{t\to\infty} V(x(t)) = 0.$$

By hypothesis on V, $\lim_{t\to 0} x(t) = x_e$ ▨

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<u>Remark</u> The above result is a prototype stability theorem in the spirit of the <u>energy method</u> in mechanics (more on this later). In the abstract setting of ordinary differential equations not necessarily hamiltonian or dissipative, the <u>Lyapunov function</u> V is a stand-in for energy functions from mechanics.

Note that $\dot{V}(x) = \frac{\partial V}{\partial x} \cdot f$. Suppose the state space can be factored into position and momentum variables, $x = (q, p) \in \mathbb{R}^n \times \mathbb{R}^n$ and

$$(3) \qquad \dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

Then $$\frac{dH}{dt} = \frac{\partial H}{\partial q} \cdot \frac{\partial H}{\partial p} + \frac{\partial H}{\partial p} \cdot \left(-\frac{\partial H}{\partial q}\right) = 0$$

If we further assume that $x_e = (q_e, p_e)$ is an equilibrium with $H(x) > H(x_e)$ for all $x \in D$ a neighborhood of x_e, then

$$V(x) = H(x) - H(x_e),$$

satisfies the hypotheses of Lyapunov's theorem allowing one to conclude x_e is a stable equilibrium.

Suppose $H(x)$ is of the form kinetic energy + potential energy,

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$$H(q,p) = \frac{1}{2}\, p \cdot M^{-1}(q)\, p + V(q),$$

$M = M^T > 0$ ← constant

where $M(q) = M^T(q) > 0$ is a mass matrix and $V(q)$ is a potential, Then (q_e, p_e) is an equilibrium of the dynamics

$$(4) \qquad \dot q = M^{-1} p, \qquad \dot p = -\frac{\partial V}{\partial q}$$

iff $p_e = 0$ and q_e is a critical point of V. If further q_e is an isolated/strict local minimum of V then (q_e, p_e) is a strict local minimum of H and hence a stable equilibrium.

The result just derived is known as the Lagrange–Dirichlet theorem and is a guiding principle in much of mechanics. It asserts the proper role of (potential) energy minimization in stability in the correct dynamical sense of Lyapunov, superseding earlier quasi-static notions e.g. due to Torricelli.

Corollary If x_e is a (stable) equilibrium of a nontrivial hamiltonian system (3), it can never be asymptotically stable.

Proof. $\exists x_0 \in \varepsilon$, $H(x_0) \neq H(x_e)$. For any trajectory beginning at x_0, $H(x(t)) = H(x_0) \neq H(x_e)$. Hence by continuity of H, $\lim_{t\to\infty} x(t) \neq x_e$ even if such a limit exists. ▨

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Hamiltonian systems of the form (4) are said to be natural mechanical systems or simple mechanical systems. A vast array of systems in classical physics, molecular dynamics, and engineering take this form, or its dissipative modification

$$\dot{q} = M^{-1} p \qquad (q,p) \in \mathbb{R}^n \times \mathbb{R}^n$$

(5)

$$\dot{p} = -\frac{\partial V}{\partial q} - R(q)\dot{q}$$

where $R(q) = R^T(q) > 0$, defines the Rayleigh dissipation function

$$\mathcal{R}(q,\dot{q}) = \tfrac{1}{2}\dot{q} \cdot R(q)\,\dot{q} \ ;$$

Along trajectories of (5),

$$\frac{dH}{dt} = -M^{-1}(q)\,p \cdot \left(R(q) M^{-1}(q)\, p\right)$$

$$\leq 0 \qquad \text{by hypothesis}$$

Suppose D is such that $(q_e, 0)$ is the only equilibrium in D (thus q_e is a critical point of V), and q_e is a strict local minimum of V. Then, can we prove asymptotic stability of $(q_e, 0)$? While stability is assured by Lagrange-Dirichlet, one cannot use H as the Lyapunov function for the asymptotic stability argument. This is because,

$$\left.\frac{dH}{dt}\right|_{(q,0)} = 0 \, .$$

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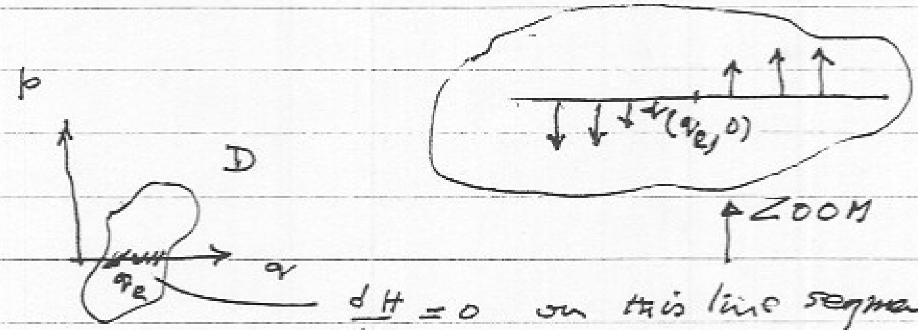

$\frac{dH}{dt} = 0$ on this line segment.

None of the points on the line segment ~~...~~ is an equilibrium (except for $(q_e, 0)$) by hypothesis on D. Does this mean we get convergence to $(q_e, 0)$ anyway? One needs a new idea, — the invariance principle of La Salle, if one insists on working with H as a possible Lyapunov function. An alternative is to fix up the Lyapunov function, i.e. add to H suitable extra terms. Both these approaches are important in solving a range of problems and we will discuss both.

<u>Example</u> (Fixing Lyapunov function) ~ damped pendulum

In equation (5) consider $n=1$, $M = 1$ constant, $V(q) = \frac{g}{l}(1-\cos(q))$, $R(q) = b > 0$

Then
$$\dot q = p$$
$$\dot p = -\frac{g}{l}\sin(q) - bp$$
$$H(q,p) = \frac{p^2}{2} + \frac{g}{l}(1-\cos(q))$$
$$\frac{dH}{dt} = -bp^2$$

$(q,p) = (0,0)$ an equilibrium

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Define

$$\tilde{H} = \frac{1}{2}(q, p)\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} + \frac{g}{l}(1-\cos(q))$$

We seek $[a_{ij}] > 0$ such that

$$\frac{d\tilde{H}}{dt} = \left(a_{11}q + a_{12}p + \frac{g}{l}\sin(q)\right)p + (a_{12}q + a_{22}p)\left(-\frac{g}{l}\sin(q) - bp\right)$$

$$= \frac{g}{l}(1 - a_{22})\,p\sin(q) - \frac{g}{l}a_{12}\,q\sin(q) + (a_{11} - a_{12}b)\,pq + (a_{12} - a_{22}b)\,p^2$$

< 0 on a suitable D.

Pick $a_{22} = 1$; $a_{11} = a_{12}b$ and $0 < a_{12} < b$. Then

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{12}b & a_{12} \\ a_{12} & 1 \end{pmatrix} > 0.$$

Further $\frac{d\tilde{H}}{dt} = -\frac{g}{l}a_{12}\,q\sin(q) - (b - a_{12})\,p^2$

< 0 on $D = \{(q,p) \mid |q| < \pi\} - \{(0,0)\}$

Thus $\tilde{H}$ is a Lyapunov function asserting asymptotic stability of $(0,0)$.

$$\tilde{H} = H + \frac{a_{12}b}{2}q^2 + a_{12}pq$$

fixup term

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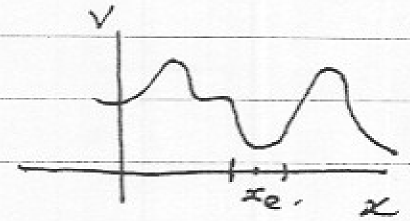

Example (Gradient Dynamics)

$V: \mathbb{R}^n \to \mathbb{R}$

$$\dot{x} = -\nabla V(x) \qquad x \in \mathbb{R}^n$$

x_e is an equilibrium iff it is a critical point of V. Suppose it is an isolated local minimum of V. From $\frac{dV}{dt} = -\nabla V \cdot \nabla V < 0$

for all $x \in B_\varepsilon(x_e) - \{x_e\}$ for $\varepsilon > 0$ small enough, we conclude asymptotic stability of x_e. ☒

The invariance principle of LaSalle is based on a fundamental result of G. D. Birkhoff, american mathematician of the first half of the 20^{th} century and a founder of the modern theory of dynamical systems.

1884–1944

Theorem <Birkhoff> If a trajectory $x(t)$, $t \geq 0$ of a dynamical system is bounded, then

$$L_+ = \omega \text{ limit set of } \{x(t) : t \geq 0\}$$

is a nonempty, compact, invariant set.

(on $\mathbb{R}^n$ with metric $d(\cdot,\cdot)$, compact $\Leftrightarrow$ closed and bounded)

Moreover $x(t) \to L_+$ as $t \to \infty$ in the sense that

$$\lim_{t \to \infty} d(x(t), L_+)$$

$$= \lim_{t\to\infty} \min_{p\in L_+} d(x(t), p)$$

$$= 0. \qquad \boxtimes$$

note, for $\dot{x} = f(x)$ $\dot{V}(x) = \frac{\partial V}{\partial x} \cdot f(x)$

We omit the proof of Birkhoff's theorem. (See for instance H. Khalil appendix A.2)

Theorem (LaSalle)

Let Ω be a compact (closed and bounded) set with the property that $\forall t \geq 0$ $x(t) \in \Omega$, whenever $x(0) \in \Omega$.

Let $V: \Omega \to \mathbb{R}$ be a C^1 function such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the ~~smallest~~ largest invariant set in E. Then every solution starting in Ω tends to M as $t \to \infty$.

Proof: Let $x(t)$ be a solution such that $x(t) \in \Omega$ $t \geq 0$. Since $\dot{V}(x) \leq 0$ in Ω, $V(x(t))$ is a monotone decreasing function of t. Since $V(x(t))$ is continuous in the compact set Ω, it is bounded below on Ω. Therefore $V(x(t)) \to a$ as $t \to \infty$. $L_+ \subset \Omega$ since Ω is closed. For any $p \in L_+$ there is a sequence $t \leq t_2 \leq \cdots \leq t_n \leq \cdots$ $t_n \to \infty$ as $n \to \infty$, such that $\lim_{n\to\infty} x(t_n) = p$.

By continuity of V, $V(p) = \lim_{n\to\infty} V(x(t_n)) = a$.

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Hence $V(x) = a$ on L_+. Since L_+ is invariant (Birkhoff), $\dot{V}(x) = 0$ on L_+. Since M is the largest invariant set $\subset E$, we get

$$L_+ \subset M \subset E \subset \Omega .$$

Since $\{x(t)\}_{t \geq 0}$ is bounded, $x(t) \to L_+$ as $t \to \infty$.

Hence $x(t) \to M$ as $t \to \infty$ $\square$

<u>Remark</u> In many problems M is much easier to determine than L_+.

In some problems one can pick Ω and V such that, x_e = equilibrium $\in \Omega$ and in fact $M = \{x_e\}$. If further x_e is stable, it is asymptotically stable by La Salle's theorem.

In (5),

$$\Omega_c = \{(q,p) : H(q,p) \leq c\}.$$

For $c > 0$ such that $H(q_e, 0) \leq c$, c small enough, Ω_c is closed & bounded.

$$E = \{(q,p) : R(q) M^{-1}(q) p = 0\} \subset \Omega_c$$

$$M = \{(q_e, 0)\}$$

proof $\to$ next page.

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$E = \{ (q, p) \in \Omega_c : p = 0 \}$

$(q(t), p(t)) \in M \quad \forall\, t \geq 0$

$\Rightarrow p(t) \equiv 0 \qquad (M \subset E)$

$\Rightarrow \dot{p}(t) \equiv 0$

$\Rightarrow$ (a) $\dot{q}(t) \equiv 0$ (from the dynamics)

$\Rightarrow q(t) \equiv q_e$ a constant

$\Rightarrow$ (b) $\dfrac{\partial V}{\partial q}(q(t)) \equiv 0$ (again from the dynamics)

$\Rightarrow \dfrac{\partial V}{\partial q}(q_e) = 0$ (from (a) above)

Thus we have shown that $(q, p) \in M$ implies $p = 0$ and q is a critical point of V. Choosing c small enough we can ensure $\{(q_e, 0)\} = M$ where q_e is a local minimum of V. Then we have shown that all trajectories starting in Ω_c $\longrightarrow (q_e, 0)$.