Alternative way to frame a curve


At \( s = 0 \), let \( T(0) \) denote the tangent vector; similarly, \( T(s) \) is the normal plane at \( s \). Pick a basis \( \{ M_1(0), M_2(0) \} \) for \( T(0) \), such that \( \{ T(0), M_1(0), M_2(0) \} \) constitutes a right-handed orthonormal triad. Our goal is to propagate this triad to \( \{ T(s), M_1(s), M_2(s) \} \) in such a way that certain natural conditions are satisfied:

- Right handedness: \( M_2(0) = T(0) \times M_1(0) \) and \( M_2(s) = T(s) \times M_1(s) \).
- \( T(s) \cdot T(s) = 1 \) implies there must exist \( k_1(s) \) and \( k_2(s) \) such that \( T(s) = k_1(s) M_1(s) + k_2(s) M_2(s) \).

We call \( k_1(s) \) and \( k_2(s) \) (natural) curvatures.
Let \( M(s) \) be any unit normal field along \( x \).

**Definition 1.** We say \( M(s) \perp T(s) \) is a relatively parallel field along \( x \) provided

\[
M'(s) = f(s) T(s)
\]
i.e. the vector \( M(s) \) turns as little as possible.

(This is the natural condition we mentioned.)

We propagate \( M_1(0), M_2(0) \) along \( x \) such that they remain relatively parallel at each \( s \).

Thus,

\[
M_1'(s) = f_1(s) T(s)
\]

\[
M_2'(s) = f_2(s) T(s)
\]

for some as yet undetermined \( f_1, f_2 \).

But \( M_1(s) \cdot T(s) \equiv 0 \) (normality),

\[
M_1'(s) \cdot T(s) = - M_1(s) \cdot T'(s)
\]

\[
= - M_1(s) \cdot (k_1(s) M_1(s) + k_2 M_2(s))
\]

\[
= - k_1(s)
\]

(m.e. \( M_1 \cdot M_1 \equiv 1 \))

and \( M_1 \cdot M_2 \equiv 0 \)

\[
\Rightarrow f_1(s) = - k_1(s)
\]

Similarly \( f_2(s) = - k_2(s) \).
**Definition 2**  
An orthonormal triad \( \{ T(s), M_1(s), M_2(s) \} \) is a \textit{relatively parallel adapted frame (RPAF)} along \( X \) provided there exist curvature functions \( k_1(s), k_2(s) \) such that

\[
T' = k_1 M_1 + k_2 M_2
\]

\[
M_1' = -k_1 T
\]

\[
M_2' = -k_2 T
\]

RPAF's are also called \textit{natural Frenet frames}.

**Theorem 3**  
Given a \( C^2 \) curve \( X \) and a choice \( M_1(0), M_2(0) \) in \( T(0) \) such that \( \{ T(0), M_1(0), M_2(0) \} \) is a right-handed orthonormal triad, there is a unique RPAF along \( X \) that agrees with the initial choice.

**Proof**  
Integrating \( M_1'(s) = -k_1(s) T(s) \) on both sides,

\[
M_1(s) = M_1(0) - \int_0^s k_1(\sigma) T(\sigma) \, d\sigma
\]

Taking the dot product with \( M_1(s) \) of both sides of

\[
T'(s) = k_1(s) M_1(s) + k_2(s) M_2(s)
\]

we get

\[
k_1(s) = T'(s) \cdot M_1(s)
\]

\[
= T'(s) \cdot M_1(0) - \int_0^s k_1(\sigma) T'(\sigma) \cdot T(\sigma) \, d\sigma
\]
Similarly:

\[ k_2(s) = x''(s) \cdot M_2(s) - \int_0^s k_2(r) \cdot x''(s) \cdot x'(r) \, dr \]

Given the curve \( x \), we have two integral equations for \( k_i, \ i = 1, 2 \). By the standard theory of Volterra integral equations, there exist unique \( k_i, \ i = 1, 2 \).

Now integrate:

\[
\frac{d}{ds} \begin{bmatrix} T(s) & M_1(s) & M_2(s) \end{bmatrix} = \begin{bmatrix} T(s) & M_1(s) & M_2(s) \end{bmatrix} \begin{bmatrix} 0 & -k_1(s) & -k_2(s) \\ k_1(s) & 0 & 0 \\ k_2(s) & 0 & 0 \end{bmatrix}
\]

Starting from \( \begin{bmatrix} T(0) & M_1(0) & M_2(0) \end{bmatrix} \in SO(3) \), to obtain a unique RPAF.

Relation to (4) The normal \( N(s) \) and binormal \( B(s) \), Frenet-Serret when defined, exist in the plane \( T_1(s) \) spanned by \( M_1(s) \) and \( M_2(s) \).

\[
N(s) = \frac{1}{k(s)} \frac{d}{ds} T(s)
\]

\[
= \frac{1}{k(s)} \left[ k_1(s) M_1(s) + k_2(s) M_2(s) \right]
\]
\[ 1 = N(s) - N(s) = \left( \frac{k_2^2(s)}{1} + \frac{k_2^2(s)}{1} \right) / \kappa(s) \]

\[ \Rightarrow \kappa(s) = \sqrt{\frac{k_2^2(s)}{1} + \frac{k_2^2(s)}{1}} \]

\[ B(s) = T(s) \times N(s) \]

\[ = T(s) \times \left( \frac{k_1(s)}{\kappa(s)} M_1(s) + \frac{k_2(s)}{\kappa(s)} M_2(s) \right) \]

\[ = \frac{1}{\kappa} \left( -k_2 M_1 + k_1 M_2 \right) \]

Torsion

\[ \gamma(s) = -B'(s) \cdot N(s) \]

\[ = -\left( -\frac{k_2}{\kappa} M_1 + \frac{k_1}{\kappa} M_2 \right)' \cdot \left( \frac{k_1}{\kappa} M_1 + \frac{k_2}{\kappa} M_2 \right) \]

\[ = \left( \frac{k_2}{\kappa} M_1' - \frac{k_1}{\kappa} M_2' + \frac{k_2}{\kappa} M_1' - \frac{k_1}{\kappa} M_2' \right) \]

\[ + k_2 M_1 \left( \frac{1}{\kappa} \right)' - k_1 M_2 \left( \frac{1}{\kappa} \right)' \cdot (\ldots) \]

\[ = \frac{k_2' k_1 - k_1' k_2}{\kappa^2} \]

\[ = \left( \tan^{-1} \left( \frac{k_2}{k_1} \right) \right)' \]

\[ = \theta' \]

where \( \theta \) = polar angle in \((k_1, k_2)\) plane, (well defined when \( k > 0 \)) also called the normal development plane.
Integrating, \[ \theta(s) = \theta(0) + \int_0^s \tau(\sigma) \, d\sigma \]

Since,

\[ N(s) = \cos(\theta(s)) M_1(s) + \sin(\theta(s)) M_2(s) \]

\[ B(s) = -\sin(\theta(s)) M_1(s) + \cos(\theta(s)) M_2(s) \]

it is clear that \( \theta(s) \) is the accumulated rotation (phase-shift) of \( \{N(s), B(s)\} \) relative to \( \{M_1(s), M_2(s)\} \).

**Definition 5**

One gets a picture of a curve \( \gamma \) in \( \mathbb{R}^3 \) by looking at its normal development \( s \mapsto (k_1(s), k_2(s)) \).

**Example 6**

Curve \( \gamma \) lies in a plane \( \mu^\perp \) perpendicular to a fixed vector \( \mu \). We do not assume \( \mu^\perp \) passes through the origin.

\[ \gamma(s) \cdot \mu = c \quad \text{a constant} \]

\[ \Rightarrow \ T \cdot \mu = 0 \]

and \( T' \cdot \mu = 0 \).

From the last equation,

\[ k_1(s) (M_1(s) \cdot \mu) + k_2(s) (M_2(s) \cdot \mu) = 0 \]
On the other hand,

\[ M'_1(s) \cdot \mu = -k_1(s) \cdot T(s) \cdot \mu = 0 \]

\[ \Rightarrow M(s) \cdot \mu = \text{constant} = a_1 \]

Similarly,

\[ M_2(s) \cdot \mu = \text{constant} = a_2 \]

Thus normal development satisfies

\[ a_1 k_1(s) + a_2 k_2(s) = 0 \]

is on a line passing through the origin. Normal development contains no information on \( s \).

Example 7. Curve \( s \mapsto \gamma(s) \) is confined to a sphere centered at \( p \) and of radius \( R > 0 \). Thus

\[ (\gamma(s) - p) \cdot (\gamma(s) - p) = R^2 \]

Differentiating,

\[ (\gamma(s) - p) \cdot \gamma'(s) = 0 \]

\[ \Rightarrow (\gamma - p) \cdot T = 0 \]
Differentiating again

\[ x'.T + (x-\mu).T' \equiv 0 \]

\[ \iff T.T + (x-\mu).T' \equiv 0 \quad \text{(def. of T)} \]

\[ \iff (x-\mu).T' \equiv -1 \quad \text{(T unit vector)} \]

\[ \iff k_1 (x-\mu).M_1 + k_2 (x-\mu).M_2 \equiv -1 \quad \text{(T' eqn)} \]

Also,

\[ M_i'.(x-\mu) = -k_i T.(x-\mu) \]

\[ \equiv 0 \quad i = 1, 2. \]

Now

\[ \begin{align*}
(M_i.(x-\mu))' &= M_i'.(x-\mu) + M_i \cdot x' \\
&= M_i'.(x-\mu) + M_i \cdot T \\
&\equiv 0
\end{align*} \]

Thus \[ M_i.(x-\mu) \equiv \text{Constant} = a_i \quad i = 1, 2. \]

Thus \[ a_1 k_1(s) + a_2 k_2(s) \equiv -1 \]

Normal development is on a line not passing through the origin, at a distance

\[ \frac{1}{\sqrt{a_1^2 + a_2^2}} \] from the origin.
We can always write

$$(\sigma - p) = ((\sigma - p) \cdot M_1) M_1 + ((\sigma - p) \cdot M_2) M_2$$

$$+ ((\sigma - p) \cdot T) T$$

where $$(\sigma - p) \cdot M_1 \equiv a_1$$; $$(\sigma - p) \cdot M_2 \equiv a_2$$

and $$(\sigma - p) \cdot T \equiv 0$$.

$$\Rightarrow R^2 = (\sigma - p) \cdot (\sigma - p)$$

$$= a_1^2 + a_2^2$$

Thus the normal development is a line not passing through the origin, at a distance $= \frac{1}{R}$ from the origin.

Normal development, being fully Euclidean invariant, contains no information about the center $p$ of the sphere.

As $R \to \infty$, sphere $\to$ plane and above line $\to$ line passing through origin.

**Remark 8** Recall that the RPAF is determined up to a choice of initial orthonormal base $\{M_1(0), M_2(0)\}$. A change of basis is simply a rotation $A$ of $\{M_1(0), M_2(0)\}$.
through an angle \( \phi \). How does this affect the curvatures \( k_1 \) and \( k_2 \)?

Let

\[
\begin{align*}
\tilde{M}_1(\phi) &= M_1(\phi) \cos(\phi) - M_2(\phi) \sin(\phi) \\
\tilde{M}_2(\phi) &= M_1(\phi) \sin(\phi) + M_2(\phi) \cos(\phi)
\end{align*}
\]

Then it can be shown by substitution in the Volterra integral equations for curvature \( k \) (see page 4) that

\[
\begin{align*}
\tilde{k}_1(s) &= \cos(\phi) k_1(s) - \sin(\phi) k_2(s) \\
\tilde{k}_2(s) &= \sin(\phi) k_1(s) + \cos(\phi) k_2(s)
\end{align*}
\]

This corresponds to a rotation by \( \phi \) in the normal development plane.

A curve \( \gamma \) determines the normal development up to such a rotation.

Example 9

Let \( s \rightarrow \gamma(s) \) be curve confined to a sphere \( (\gamma - \nu) \cdot (\gamma - \nu) = R^2 \), centered at \( \nu \in \mathbb{R}^3 \), of radius \( R \).

For \( s = 0 \) pick \( \gamma(0) = \frac{\gamma(0) - \nu}{R} \).

Clearly \( \tilde{M}_1(\phi) \in T(0) \) by hypothesis.
We let $M_2(o) = T(o) \times M_1(o)$ to make up the initial, right-handed orthonormal frame $\{ T(o), M_1(o), M_2(o) \}$.

Now $a_1 = (\gamma(o) - p) \cdot M_1(o)$

$$= (\gamma(o) - p) \cdot M_1(o)$$

$$= (\gamma(o) - p) \cdot \frac{(\gamma(o) - p)}{R}$$

$$= R$$

$a_2 = (\gamma(o) - p) \cdot M_2(o)$

$$= (\gamma(o) - p) \cdot M_2(o)$$

$$= (\gamma(o) - p) \cdot (T(o) \times \frac{(\gamma(o) - p)}{R})$$

$$= 0$$

The normal development equation becomes

$$-1 \equiv a_1 k_1(o) + a_2 k_2(o)$$

$$\equiv R k_1(o) + 0$$

$$\Rightarrow k_1(o) \equiv -1/R$$
Thus the evolution equation of an RPAF for a curve confined to a sphere of radius R (centered at \( p \)) can always be taken to be of the form:

\[
\begin{bmatrix}
T' & M_1' & M_2'
\end{bmatrix} =
\begin{bmatrix}
0 & V_R & -k_2(s) \\
-V_R & 0 & 0 \\
k_2(s) & 0 & 0
\end{bmatrix}
\]

with,

\[M_1(0) = \frac{\mathbf{x}(0) - p}{R}; \quad M_2(0) = T(0) \times M_1(0)\]

Since \( (\mathbf{x}(s) - p). M_1(s) = R \),

\[\frac{\mathbf{x}(s) - p}{R}. M_1(s) = 1\]

Let \( M_1(s) = \frac{\mathbf{x}(s) - p}{R} + \mathbf{s}(s) \).

Then \( \frac{\mathbf{x}(s) - p}{R}. \frac{\mathbf{x}(s) - p}{R} + \frac{\mathbf{x}(s) - p}{R}. \mathbf{s}(s) = 1 \)

\[\Rightarrow 1 + \frac{\mathbf{s}(s) - p}{R}. \frac{\mathbf{s}(s)}{R} = 1\]

\[\Rightarrow \frac{\mathbf{s}(s) - p}{R}. \mathbf{s}(s) = 0\]

Then \( M_1(s). M_1(s) = \left(\frac{\mathbf{x}(s) - p}{R} + \mathbf{s}(s)\right) \cdot \left(\frac{\mathbf{s}(s) - p}{R} + \mathbf{s}(s)\right)\)

\[= 1 + \mathbf{s}(s) \cdot \mathbf{s}(s)\]

But \( M_1(s) \) is a unit vector for each \( s \).

Hence \( \mathbf{s}(s) \cdot \mathbf{s}(s) \equiv 0 \Rightarrow \mathbf{s}(s) \equiv 0\)

\[\Rightarrow M_1(s) \equiv (\mathbf{x}(s) - p)/R\]
Thus, if \( M_1(0) = \frac{8(0) - R}{R} \), \( M_1(t) = \frac{8(t) - R}{R} \) vs.

Thus, \( M_1 \) is the outward normal always.