Center Manifold Theorem and Reduction
(based on H. Khalil, 2nd edition)

The critical cases of hyperstability, i.e., the systems for which the indirect method is inconclusive due to the presence of eigenvalues on the imaginary axis can be examined via the center manifold theorem.

Consider the nonlinear system
\[ \dot{y} = A_1 y + g_1(y, z) \]
\[ \dot{z} = A_2 z + g_2(y, z) \]
where spectrum \((A_1) \subseteq \text{imaginary axis}, \)
\( \text{spectrum } (A_2) \subseteq \mathbb{C} \text{ and } g_1 \) and \(g_2\) are both \(C^2\), \(g_i(0,0) = 0\), \( \frac{\partial g_i}{\partial y}(0,0) = \frac{\partial g_i}{\partial z}(0,0) = 0 \) \(i = 1,2\).

Then there exists \(\delta > 0\) and \(C\) function \(h(y)\) defined for all \(y\) satisfying \( \|y\| < \delta\), such that \(z = h(y)\) is a center manifold, i.e.,
\[ h(0) = 0, \frac{\partial h}{\partial y}(0) = 0 \text{ and } z = h(y) \text{ is an invariant manifold of } (\ast). \]

Proof: The proof of this existence theorem is an application of the contraction mapping fixed point theorem on the space of maps \(f : \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}\) equipped with the sup norm.

The change of variables

\[
\begin{pmatrix}
  y \\
  z
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  y \\
  u
\end{pmatrix}
= \begin{pmatrix}
  y \\
  z - h(y)
\end{pmatrix}
\]

maps the center manifold so constructed into the set \( w = 0 \) and invariance implies \( \dot{w} = 0 \)

\[
\Rightarrow \quad \dot{z} - \frac{\partial h}{\partial y} \dot{y} = 0
\]

\[
\Rightarrow \quad A^y(y) + g_1(y, h(y)) - \left( \frac{\partial h}{\partial y} \right) (A, y + g_1(y, h(y))) = 0
\]

function \( h \) defining the tensor \( A \).

Thus the center manifold satisfies the partial differential equation

\[
A^y(y) + g_1(y, h(y)) = \frac{\partial h}{\partial y} (A, y + g_1(y, h(y)))
\]

The dynamics in the \( y, w \) coordinates take the form (or with a bit of adding/subtracting the

\[
\text{(**) }
\]

\[
\dot{y} = A_1 y + \frac{1}{2} (g_2(y, h(y))) + N_2(y, w)
\]

\[
\dot{w} = A_2 w + \frac{1}{2} N_2(y, w)
\]

where

\[
N_1(y, w) = g_1(y, w + h(y)) - g_1(y, h(y))
\]

\[
N_2(y, w) = \frac{\partial^2}{\partial y^2} (g_2(y, w + h(y)) - g_2(y, h(y))) - \frac{\partial h}{\partial y} N_1(y, w)
\]
We have used here the partial differential equation satisfied by the center manifold:

It is easy to verify that $N_1$ and $N_2$ are both $C^2$ (since $g_i$ are $C^2$), and

$$N_i(y, 0) = 0, \quad \frac{\partial N_i}{\partial w}(y, 0) = 0, \quad i = 1, 2.$$ 

Hence, in a domain $\|w\|_2 < r$, $N_1$ and $N_2$ satisfy

$$\|N_i(y, w)\|_2 \leq k_i \|w\|_2, \quad i = 1, 2,$$

where the positive constants $k_i$ could be made arbitrarily small by choosing $p$ small enough.

Restricted to the invariant manifold $w = 0$, the dynamics $(\dot{x}, x)$ takes the form

$$\dot{y} = A y + g_1(y, y, g_2),$$

which we call the reduced system. The equilibrium $(0, 0)$ of the system $(\dot{x}, x)$ projects to the equilibrium $0$ of the reduced system $(\dot{y}, y)$. The main result of interest to us is that stability of the equilibrium of the un-reduced system can be determined from those of the reduced system.
Theorem (Reduction and stability).

If the origin $y = 0$ of the reduced system (1) is asymptotically stable (respectively unstable) then the origin of the full system (**) is asymptotically stable (respectively unstable).

Proof: First we note that if the origin of the reduced system is unstable, then any solution $y(t)$ of the reduced system, no matter how small $\|y(0)\|$ is, leaves the set $\{y : \|y\| < \varepsilon\}$ eventually. Hence the solution $(y(t), 0)$ of the full unreduced system leaves the set $\{(y,w) : \|y,w\| < \varepsilon\}$ eventually. Thus $(0, 0)$ is an unstable equilibrium of the unreduced system.

(Aside: we have also shown that stability of the origin of the unreduced system implies stability of the origin of the reduced system.)

Suppose that the origin of the reduced system is asymptotically stable. (The reduced system is autonomous. So one can specialize to the autonomous setting; one of the converse Lyapunov theorems — in fact there is one best suited to this. See Hassani Khalil.

(Second Edition Theorem 3.14) (Third edition, Theorem 4.16)

Then there is a $C^1$ function $V$ such that
V(y) is positive definite (i.e. $V(y) \geq 0, \forall y$) for a suitable class $\mathcal{K}$ function and satisfies the following inequalities in a neighborhood of the origin,

$$\frac{\partial V}{\partial y} \left( A_2 y + g_1(y, h(y)) \right) \leq -\alpha_3 \|y\|_2$$

$$\|\frac{\partial V}{\partial y}\|_2 \leq \alpha_4 \|y\|_2 \leq k$$

where $\alpha_3$ and $\alpha_4$ are class $\mathcal{K}$ functions.

On the other hand, since $A_2$ is a Hurwitz matrix (spectrum $(A_2) \subseteq \mathbb{C}^-$), the Lyapunov equation

$$A_2^T P + PA_2 = -\frac{1}{k} I$$

has a unique positive definite solution $P = P^T$.

Consider $W(y, w) = V(y) + \sqrt{w^T P w}$ as a Lyapunov function candidate for the full system (**) as well. It is clearly a positive definite function. Along trajectories of (**)

$$W(y, w) = \frac{\partial V}{\partial y} \left( A_2 y + g_1(y, h(y)) + N_1(y, w) \right)$$

$$+ \frac{1}{2 \sqrt{w^T P w}} \left( (w^T (A_2^T P + PA_2) w + 2w^T N_1) \right)$$

$$\leq -\alpha_3 \|y\|_2 + kk_1 \|w\|_2$$
\[
\frac{\|W\|_2}{\sqrt{\lambda_{\text{max}}(P)}} + k_2 \frac{\lambda_{\text{max}}(P)}{\sqrt{\lambda_{\text{min}}(P)}} \|W\|_2
\]

At this step we use \(\|W\|_2 \leq k_1 \|W\|_2\),
\[
\|\frac{\partial V}{\partial y}\|_2 \leq k, \quad \lambda_{\min}(P) \|W\|_2^2 \leq W^T P W \leq \lambda_{\text{max}}(P) \|W\|_2^2
\]

Hence,
\[
W(y, w) \leq -\sigma_3 (\|y\|_2) - \frac{\|W\|_2}{4 \sqrt{\lambda_{\text{max}}(P)}} - \left[ \frac{1}{4 \sqrt{\lambda_{\text{max}}(P)}} - k k_1 - k_2 \frac{\lambda_{\text{max}}(P)}{\sqrt{\lambda_{\min}(P)}} \right] \|W\|_2^2
\]

Since \(k_1\) can be made arbitrarily small by restricting the domain around the origin (i.e., by picking \(\|W\|_2 < \rho\) and \(\rho\) small enough), we can choose \(\rho\) to ensure that,
\[
\frac{1}{4 \sqrt{\lambda_{\text{max}}(P)}} - k k_1 - k_2 \frac{\lambda_{\text{max}}(P)}{\sqrt{\lambda_{\text{min}}(P)}} > 0
\]
Hence
\[ \dot{W}(y, w) \leq -\delta_3 (\|y\|_2) - \frac{1}{2} \left\| \frac{d}{d\gamma} \right\|_{\text{max}} (P) \|W\|_2 \]
which implies asymptotic stability of the origin in the full system (**).

Corollary 1
If the origin \( y = 0 \) of the reduced system is stable, and there is a positive definite \( C^1 \) Lyapunov function \( V(y) \) such that
\[ \frac{dV}{dy} (A, y + g, (y, h(y))) \leq 0 \]
in some neighborhood of \( y = 0 \), then the origin of the full system is stable.

Corollary 2
The origin of the full system (***) is asymptotically stable if the origin of the reduced system is asymptotically stable.
The reduction theorem — a User’s Manual

\( h \) satisfies the first order partial differential equation

\[
W(h(y)) = \frac{dh}{dy} \left( A_1 y + g_1(y, h(y)) \right) - A_2 \frac{d^2 h}{dy^2} - g_2(y, h(y))
\]

\[ = 0 \]

with \( h(0) = 0 \) and \( \frac{dh}{dy}(0) = 0 \)

Directly trying to compute the center manifold \( h() \) is difficult. If we could approximate \( h() \) and make statements based on an approximate reduced system then it is possible to use the reduction theorem as a practical tool.

Theorem If a continuously differentiable function \( \phi(y) \) with \( \phi(0) = 0 \) and \( \frac{d\phi}{dy}(0) = 0 \) can be found such that

\[
W(\phi(y)) = O(\|y\|^p) \quad \text{for some } p > 1
\]

then for sufficiently small \( \|y\| \),
\[ P(y) - \phi(y) = O(\|y\|^{p+1}) \]

and the reduced system can be represented as

\[
\dot{y} = A_1 y + g_1(y, \phi(y)) + O(\|y\|^{p+1})
\]

How do we use this result?
The main idea behind center manifold reduction approach to stability assessment is to approximate the center manifold.

If an approximation at a certain order does not work (i.e., it is inconclusive on stability), then try a higher order approximation. The approximations to $h(y)$ are constructed by seeking $\phi(y)$ such that

$$N(\phi(y)) = O(y^{1+})$$

where,

$$\phi(y) = b_2 y[y^2] + b_3 y[y^3] + \cdots + b_k y[y^k]$$

(Read $\phi(0) = 0$ & $\phi'(0) = 0$)

(The $y[y^k]$ notation is explained in Chapter 5 of Sanyal's book - c.f. Centerman expansion/linearization, and $b_k$ are appropriate matrices satisfying the condition $N(\phi(y)) = O(y^{1+})$). In the scalar case $y[y^k] = y^k$. Substitute the $\phi$ into the formula for the reduced system

$$\dot{y} = A y + f(y, \phi(y)) + O(y^{1+})$$
and assess stability of the origin.

If this step is inconclusive try a higher \( p \).

**Example 1**

\[
\begin{align*}
  \dot{x}_1 &= x_2 & a \neq 0 \\
  \dot{x}_2 &= -x_2 + ax_1^2 + bx_1x_2 \\
  A &= \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad \text{Thus} \quad k = 1. \\
  \text{Define} \quad y &= x_1 + x_2 \\ z &= -x_2 \\
  \implies x_1 &= y + z \\
  x_2 &= -z \\
  \Rightarrow y &= \frac{a(y+z)^2 - b(y^2+z^2)}{9(y+z)} \\
  \frac{dz}{dy} &= -\frac{z}{y} = -\frac{a(y+z)^2 + b(y^2+z^2)}{9(y+z)} \\
  A_1 &= 0; \quad A_2 = -1 \\
  N|_{(y(\eta),z(\eta))} &= \frac{\partial^2}{\partial y^2} (a(y + h(y))^2 - b(y + h(y) + h''(y))) \\
  &\quad + h(y) + a(y + h(y)) - b(y + h(y) + h''(y)) \\
  \dot{h}(o) &= 0, \quad \ddot{h}(o) = 0.
\end{align*}
\]
Consider \( \phi = 2 \)

\[
\phi(y) = h_2 y^2
\]

\[
W(\phi(y)) = 2 h_2 y (a (y + h_2 y^2)^2 - b (y h_2 y^2 + \frac{1}{2} y^4)) \\
+ h_2 y^2 + a (y + h_2 y^2)^2 - b (y h_2 y^2 + \frac{1}{2} y^4)
\]

\[
= O(1y^2),
\]

for any choice of \( h_2 \). One might try for simplicity \( h_2 = 0 \).

Then the reduced system is

\[
y = ay^2 + O(1y^3)
\]

For \( a > 0 \), the origin of the reduced system is unstable \( \Rightarrow \) instability for the full system.

Here we used the property: For a scalar system \( \dot{y} = ay^k + O(1y^{k+1}) \), \( k \) a positive integer,

- 0 is asymptotically stable if \( k \) odd and \( a < 0 \)
- 0 is unstable if \( k \) odd and \( a > 0 \)
- or if \( k \) even and \( a \neq 0 \)
Example 2

\[ y' = y^2 \]
\[ z' = -z + ay^2 \]

Center manifold equation

\[ N(h(y)) = \frac{\partial h}{\partial y} \cdot (y, h(y)) + h(y) - ay^2 = 0 \]

\( h(0) = z(0) = 0 \cdot \]

\( \phi(y) = h_2 y^2 \]

\[ N(\phi(y)) = \frac{\partial h}{\partial y} \cdot (y, h_2 y^2) + h_2 y^2 - ay^2 \]

\[ = 0(1y^4) \quad \text{for any } h_2 \]

Let \( h_2 = 0 \cdot \]

Reduced equation

\[ y' = y h_2 y^2 + 0(1y^3) \]

\[ = 0(1y^3) \]

which is inconclusive on stability

So try \( \phi(y) = h_2 y^2 + h_3 y^3 \)

Then \[ N(\phi(y)) = (2h_2 y + 3h_3 y^2) (y (h_2 y^2 + h_3 y^3)) + (2y^2 + h_3 y^3) - ay^2 \]
\[ \frac{14}{14} \]

In that case the reduced system is
\[
\dot{y} = y \left( 2y^2 + b_3 y^3 \right) + O(141^4)
\]
\[ = ay^3 + O(141^4) \]
for which \(0\) is asymptotically stable provided \(a < 0\)
\(0\) is unstable if \(a > 0\)

Consequently the origin is asymptotically stable for the full system if \(a < 0\)
unstable for the full system if \(a > 0\)

Example 3

This example illustrates the case where the center manifold dimension is \(1\).

\[
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}
+ \begin{pmatrix}
-y_1^3 \\
y_2^3 + y_2^2
\end{pmatrix}
\]
\[ \dot{z} = -z + (y_1^3 - 3y_1^2 + 3y_1y_2^2) \]
\[ \phi(y) = 0 \Rightarrow N(\phi(y)) = O(141^3) \]
and
\[
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix} =
\begin{pmatrix}
-y_1^3 + y_2^2 \\
y_2^3 + y_2^2
\end{pmatrix} + O(141^4) \]
Let \( V(y) = \frac{1}{2} (y_1^2 + y_2^2) \).

\[
\dot{V} = -y_1 y_1^4 - y_2 y_2^4 + y^T \nabla V(y) \\
\leq -\|y\|_2^4 + k \|y\|_2^5
\]

in some neighborhood of the origin where \( k > 0 \). Hence

\[
\dot{V} \leq \frac{1}{2} \|y\|_2^4 \text{ for } \|y\|_2 < \frac{1}{ak}
\]

\[\Rightarrow 0 \text{ is asymptotically stable} \]