Lecture 10

Nonlinear System and Feedback

Consider the system

$$\dot{x} = f(x, u)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. Suppose $f(0, 0) = 0$ and $f \in C^1$. Let

$$A = \left( \frac{\partial f}{\partial x} \right)_{(0, 0)}$$

and let

$$B = \left( \frac{\partial f}{\partial u} \right)_{(0, 0)}$$

Hypothesis 1

Let $K$ be such that

$$\text{spectrum } (A + BK) \subseteq \mathbb{C}$$

Consider the closed loop system

$$\dot{x} = f(x, u)$$

$$u = Kx .$$

Thus

$$\dot{x} = \tilde{f}(x) = f(x, Kx) .$$

Clearly $\tilde{f}(0) = 0$. The linearization of the closed loop system at the origin is

$$\tilde{z} = \tilde{A} \tilde{z} ,$$

where

$$\tilde{A} = \left( \frac{\partial \tilde{f}}{\partial x} \right) .$$
But \( \left( \frac{\partial f}{\partial x} \right) \bigg|_{x=0} = \left. \frac{\partial f}{\partial x}(x, kx) \right|_{x=0} \)

\[ = \left. \frac{\partial f}{\partial x} \right|_{x=0} + \left. D_x f \cdot k \right|_{x=0} \quad \text{(chain rule)} \]

\[ = (A + BK). \]

By hypothesis 1, and the indirect method of Lyapunov, the origin is an asymptotically stable equilibrium of the closed-loop system.

Remark: A sufficient condition for hypothesis 1 to hold is that the pair \([A, B]\) is controllable. \(<\text{Recall: the eigenvalue/pole placement theorem}>\)

We see that \( u = Kx \) a linear feedback law, can be stabilizing. The region of attraction may be estimated by

(i) solving \((A + BK)^T P + P (A + BK) = -Q\) for a \( Q = Q^T > 0 \)

(ii) let \( g(x) = f(x, kx) - (A + BK)x \)
and observe that $||g(x)|| \leq 5 \times 10^{-12}$ and $||x||$ can be made arbitrarily small by choosing $r$ small.

This is the argument discussed in pages 4 & 5 of Lecture Note 6 (point iii).

By (d), $y_{(0)}$ may be too close for practical purposes. The approach here is to stabilize the system to exactly linearize $\epsilon$. Any equilibria do not have the existence of an additional linear feedback. Thus, the system is destabilized.
Definition

Let \( y = f(y) + G(y)u \) where
\[
\begin{align*}
  f(0) &= 0, \\
  G(y) &= \begin{bmatrix} g_1(y) & \ldots & g_m(y) \end{bmatrix} \\
  g_i(0) &= 0.
\end{align*}
\]

We say that this system is \underline{exact state feedback linearizable} if there exists an open \( \mathbb{R}^n \rightarrow \mathbb{R}^n \) map \( T : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n \) of \( U \), \( T \) is \( C^\infty \) and invertible, \( T^{-1} \) exists and is \( C^\infty \), and functions \( \alpha(x) \) and \( \beta(x) \) such that under the change of coordinates by \( T \),

\[
x = T(y)
\]

satisfies
\[
\dot{x} = Ax + B \beta^{-1}(x) [u - \alpha(x)]
\]

\( \equiv Ax + Bv \)

where \( v \equiv \beta^{-1}(x) \cdot [u - \alpha(x)] \)

and \( [A, B] \) is controllable.

We then seek \( \beta(x) \) such that \( \beta(T(y)) \) is an invertible \( m \times m \) matrix at every \( y \).
The system outside the dotted line is linear.

By chain rule

\[
\dot{x} = \frac{\partial T}{\partial y} \dot{y}
\]

\[
= \frac{\partial T}{\partial y} \left( f(y) + g(y) u \right)
\]

\[
= A \dot{x} + B \beta^{-1}(x) \left[ u - \alpha(x) \right]
\]

\[
= AT(y) + B \beta^{-1}(T(y)) \left[ u - \alpha(T(y)) \right]
\]

\[ y \in U. \]

Set \( u \equiv 0 \)

\[
\Rightarrow \begin{cases} 
\frac{\partial T}{\partial y} f(y) = AT(y) - B [F(T(y))]^{-1} \alpha(T(y)) \\
\frac{\partial T}{\partial y} g(y) = B [F(T(y))]^{-1}
\end{cases}
\]
Consider the single input case \((m = 1)\) with canonical form \( A = A_c, \quad B = B_c \) and \( G_c = g \)

\[
A_c = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}, \quad B_c = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]

Then the conditions above on \( T \) take the form

\[ \begin{align*}
(1a) \quad & \frac{d}{dy} T_1 \cdot f(y) = T_2 \cdot g(y) \\
(2a) \quad & \frac{d}{dy} T_2 \cdot f(y) = T_3 \cdot g(y) \\
& \vdots \\
(h-1)a \quad & \frac{d}{dy} T_{h-1} \cdot f(y) = T_h \cdot g(y) \\
(ha) \quad & \frac{d}{dy} T_h \cdot f(y) = -\frac{d'}{\beta} \cdot \frac{d}{dy} T_h \cdot g(y) = \frac{1}{\beta} \\
\end{align*} \]

(assuming \( \beta \neq 0 \))
Define the operation (Lie derivative)

\[ L_f h = \frac{2h}{2x} f \]

where \( h \) is a scalar function and \( f \) is a vector field.

Define \( L_f h = h \)

and

\[ L^{k+1}_f h = L_f (L_k f h) \]

With these definitions we have relations,

\[ T_k = L f T_1 \quad k = 2, 3, \ldots, n \]

And we have the equations

\[ \begin{bmatrix} L_k f T_1 \\ f \end{bmatrix} = 0 \quad k = 0, 1, 2, \ldots, (n-2) \]

If we can solve these equations for \( T_1 \), then by using the recursion in above we can define \( T_k, k = 2, \ldots, n \) and

\[ P = \left( L_g T_n \right)^{-1} \]

\[ \delta g \beta = \text{identical to} \quad \text{then} \quad \alpha = -\frac{L_f T_n}{L_g T_n} \]
What about solvability of $\mathcal{L}_{T_1}$?

Define

$$\text{ad}_f g = \left( \frac{\partial g}{\partial x} \right)_f - \left( \frac{\partial f}{\partial x} \right)_g$$

and also

$$\text{ad}_f g = \text{ad}_g f$$

Therefore

$$\text{ad}^{k+1}_f g = \text{ad}_f \left( \text{ad}_g^k f \right)$$

**Theorem**

There exists, (locally in a suitable neighborhood of \( \theta \)) a function $T_1$, s.t.

$$L \mathcal{L}_{T_1} g = 0 \quad \text{for} \quad k = 0, 1, 2, \ldots, (n-2)$$

**Proof**

(i) $\{g, \text{ad}_f g, \text{ad}_f^2 g, \ldots, \text{ad}_f^{n-1} g\}$ is a set of linearly independent vector fields.

(ii) $\{g, \text{ad}_f g, \text{ad}_f^2 g, \ldots, \text{ad}_f^{n-2} g\}$ is a set of vector fields satisfying the involutivity property.
\[ p(x), q(x) \in \text{this set} \]

\[ \Rightarrow \left( \frac{\partial q}{\partial x} \right) p(x) - \left( \frac{\partial p}{\partial x} \right) q(x) \text{ also belong to this set} \]

Remark: This existence result is a consequence of Frobenius' Theorem in differential geometry.

Example:

\[ \dot{y} = f(y) + g(y) u \]

\[ f(y) = \begin{bmatrix} y_2 \\ -a \sin(y_4) - b(y_1 - y_3) \\ y_4 \\ c(y_1 - y_3) \end{bmatrix} \]

\[ g(y) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ p \neq 0 \quad \forall \theta \]
\[ L_y T_1 = 0 \iff \frac{\partial T_1}{\partial y_4} = 0 \]
\[ \Rightarrow T_1 \text{ independent of } y_4. \]

\[ T_2 = L_f^{-1} T_1 \]
\[ = \frac{\partial T_1}{\partial y_1} y_2 + \frac{\partial T_1}{\partial y_2} \left( -a \sin y_1 - b (y_1 - y_3) \right) \]
\[ + \frac{\partial T_1}{\partial y_3} y_4 \]
\[ L_y T_2 = 0 \iff \frac{\partial T_1}{\partial y_3} = 0 \]
\[ \Rightarrow T_1 \text{ independent of } y_3 \]
\[ \Rightarrow T_2 = \frac{\partial T_1}{\partial y_1} y_2 + \frac{\partial T_1}{\partial y_2} \left( -a \sin y_1 - b (y_1 - y_3) \right) \]

\[ T_3 = L_f^{-1} T_2 \]
\[ = \frac{\partial T_2}{\partial y_1} y_2 + \frac{\partial T_2}{\partial y_2} \left( -a \sin y_1 - b (y_1 - y_3) \right) + \frac{\partial T_2}{\partial y_3} y_4 \]
\[ L_y T_3 = 0 \iff \frac{\partial T_3}{\partial y_4} = 0 \iff \frac{\partial T_2}{\partial y_3} = 0 \]

\[ a, b, c > 0. \]
\[ b \frac{dT_1}{dy_2} = 0 \implies \frac{dT_1}{dy_2} = 0 \]

So \( T_1 \) is independent of \( y_2 \)

So \( T_1 = T_1(y_1) \)

Pick \( T_1(y_1) = y_1 \) \( (\text{trivial}) \).

Then \( x_1 = y_1 \) \( /\!\!/ \)

\[ x_1 = y_1 = y_2 \quad (\text{from model}) \]

But \( x_1 = x_2 \) \( (\text{linear system}) \)

So \( x_2 = T_2(y_1) = y_2 \) \( /\!\!/ \)

\[ x_3 = T_3(y_2) = \dot{x}_2 = y_2 \]

\[ = -a \sin y_1 - b (y_1 - y_3) \quad (\text{non-linear model}) \]

\[ x_4 = T_4(y) = \dot{x}_3 = -a y_1 \cos y_1 \]

\[ - b (y_1 - y_3) \]

\[ = -a y_2 \cos (y_1) - b (y_2 - y_4) \]

Check \( \beta \) & \( \phi \) are well defined \( /\!\!/ \)