The understanding of systems from an external stimulus-response or input-output point of view has a long history, pre-dating the infusion of state-space or internal descriptions. It is the natural thing to consider in exploring a wide variety of complex systems (from economics, biology as well as the world of technology). In some settings, definitions and theorems in the state-space point-of-view lead to corresponding results in the external point-of-view. The converse is not the case, without additional hypotheses.

To illustrate:

Consider a linear time varying system

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) \]
\[ y(t) = C(t)x(t) \]

Assume that

(i) the transition matrix is defined
\[ \dot{\Phi}(t, t_0) = A(t) \Phi(t, t_0) \]

\[ \Phi(t_0, t_0) = I \]

satisfies \[ ||\Phi(t, t_0)|| \leq m e^{-k(t-t_0)} \]

\( t > t_0 \), and some \( k > 0, \ m > 0 \).

(Thus the system \( \dot{x}(t) = A(t)x(t) \)

(Has finite exponential stability of the zero solution)

(We call this internal stability)

\( (i) \quad ||c(t)|| \leq c \quad ||B(t)|| \leq b \)

\( t > t_0 \).

The variation of constants formula tells us that

\[ y(t) = c(t) \Phi(t, t_0) x_0 + \int_{t_0}^{t} c(t) \Phi(t, s) B(s) u(s) \, ds \]

\[ \Rightarrow \quad ||y(t)|| \leq cm e^{-k(t-t_0)} ||x_0|| + \frac{cbm}{k} \int_{t_0}^{t} e^{-k(t-s)} ||u|| \, ds \]

where we assume bounded inputs:

\[ ||u(t)|| \leq \beta \quad \Rightarrow \quad ||u(t)|| \leq \gamma \]

\( t > t_0 \).

\[ \Rightarrow \quad ||y(t)|| \leq \beta + \gamma ||u|| \]

where \( \beta = cm ||x_0|| \) & \( \gamma = \frac{cbm}{k} \)

Internal Stability + Bounded Inputs \( \Rightarrow \) Bounded Outputs
The property of bounded inputs always giving rise to bounded outputs is a type of external stability. As can be seen from the example below:

\[
\begin{align*}
\dot{x}_1 &= -x_1 + u \\
\dot{x}_2 &= x_1^2 + x_2^2 \\
y &= x_1
\end{align*}
\]

external stability \(\Rightarrow\) internal stability (of the zero solution)

We would like to state and prove certain basic notions & theorems of external stability, connect them to interesting physical properties of systems and establish ties to notions of internal \(\ddot{\text{stability}}\). The initial steps in this direction include:

(a) proper definitions of function spaces of input and output signals

(b) concepts of causality, feedback, well-posedness and passivity
The signals applicable in the present context cannot be of the finite energy over the infinite time interval \([0, \infty)\). Think of ramp signals.

**Definition**
The truncation operator \(\cdot)_T\) on functions on \([0, \infty)\) is defined by

\[
X_T(t) = \begin{cases} 
X(t) & t \leq T \\
0 & t > T
\end{cases}
\]

for \(T > 0\).

**Definition**
The space \(L^p_\infty\) is defined by

\[
L^p_\infty = \left\{ X : [0, \infty) \to \mathbb{R} \mid X \in L^p, \forall T > 0 \right\}
\]

**Example:**
\[
X(t) = t, \quad t > 0
\]
\[
X(\cdot) \notin L^p \text{ for any } p \in [1, \infty)
\]

But \(X_T \in L^p, \forall T > 0\).

**Lemma:** For each \(p \in [1, \infty]\), the set \(L^p_\infty([0, \infty])\) is a linear space. If \(p \in [1, \infty]\) and \(f \in L^p_\infty([0, \infty])\), then

(i) \(\|X_T\|\) is a non-decreasing function of \(T\)

(ii) \(f \in L^1([0, \infty]) \iff \) there exists a finite constant \(M\) such that \(\|f_T\| \leq M, \forall T > 0\). In that case,

\[
\|f\|_p = \lim_{T \to \infty} \|f_T\|_p
\]

[PROOF] [EXERCISE]
Remark: $L^p([0,\infty))$ itself does not carry a norm that agrees with the norm on $L^p([0,\infty))$ when restricted to that subspace.

$L^p = L^p \times L^p \times \ldots \times L^p$

$r$ times

i.e. each function $f \in L^p$ is characterized by each component $f_i \in L^p$. Similarly for $L^q$.

Definition (Causality)

$F: L^m_{pe} \to L^q_{pe}$ is said to be a causal map/system if

$(F(u))(t) = (F(u_{-t}))(t) \quad \forall t \geq 0$ and $u \in L^m_{pe}$

Lemma: A map/system $F: L^m_{pe} \to L^q_{pe}$ is causal if whenever $u_1, u_2 \in L^m_{pe}$ and $(u_1)_T = (u_2)_T$ for some $t < \infty$, we have $(F(u_1))(t) = (F(u_2))(t)$.

Proof ($\Rightarrow$) Suppose $F$ satisfies the condition in the statement. Let $u \in L^m_{pe}$. Let $T < \infty$ be arbitrary.

Then $(u)(t) = (u_T)(t)$. By hypothesis,

$(F(u))(t) = (F(u_T))(t)$.

Since $T$ is arbitrary, we have established causality.

($\Leftarrow$) Assume $F$ is causal.
Let $u_1, u_2 \in L^p$ be such that for some $T > 0$, $(u_1)_T = (u_2)_T$.

Now $(F(u_1))_T = (F(u_2))_T = (F(u_2)_T)_T = (F(u_2))_T$.

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**Stability in the external source**

**Definition (1)** A map/system $F : L^p \to L^q$

is said to be stable if there exist finite constants $\alpha, \beta > 0$ such that

$$
\|F(u)_T\| \leq \alpha \|u\|_T + \beta
$$

for $u \in L^p$ and $T > 0$.

**Definition (1')** A map/system $F : L^p \to L^q$

is said to be stable if

(i) $F(u) \in L^q$ whenever $u \in L^p$, and in that case

(ii) there exist constants $\alpha, \beta > 0$ s.t.

$$
\|F(u)\| \leq \alpha \|u\| + \beta
$$

for $u \in L^p$.

**Remark** The two definitions (1) & (1') are equivalent.
Small Gain Theorem

\[ u_1 + e_1 \rightarrow H_1 \rightarrow y_1 \]

H_2 \rightarrow e_2 \rightarrow u_2

Assume.

(H1) The maps \( H_i : L_{pe}^m \rightarrow L_{pe}^q \)

\[ H_1 : L_{pe}^m \rightarrow L_{pe}^q \]

\[ H_2 : H_{pe}^q \rightarrow L_{pe}^m \]

are causal.

(H2) Suppose \( H_i \) are stable with gains \( x_i \)

and offsets \( \beta_i \) satisfying

\[ \| H_i (u) \| \leq x_i \| u \| + \beta_i \]

\( i = 1, 2 \)

(H3) For every pair of inputs \( u_1 \in L_{pe}^m \)

and \( u_2 \in L_{pe}^q \), there exist unique outputs

\( e_1 \in L_{pe}^m \) and \( e_2 \in L_{pe}^q \) [Well-posedness]

If further \( x_1 x_2 < 1 \), then

(a) \( u_1 \in L_{pe}^m \) and \( u_2 \in L_{pe}^q \)

\[ \| e_1 \| \leq \frac{1}{1 - x_1 x_2} \left( \| u_1 \| + \frac{x_1}{2} \| u_2 \| + \beta_1 + \beta_2 \right) \]

\[ \| e_2 \| \leq \frac{1}{1 - x_1 x_2} \left( \| u_2 \| + x_1 \| u_1 \| + \beta_1 + \beta_2 \right) \]
and (b) if \( u_1 \in L^p \) and \( u_2 \in L^q \), then
\[ e_1, y_2 \in L^p \] and \( e_2, y_1 \in L^q \), and the norms
of \( e_1 \) and \( e_2 \) are bounded above by the r.h.s.
in (a) with nontruncated functions.

**Proof**

Below, we will use causality freely to write \( \| F(u) + 1 \| = \| F(u) \| + 1 \) as needed.

By hypothesis (a2) we can solve uniquely for
\[
\begin{align*}
e_1 &= u_1 - (h_2(e_2) + )T, \quad \text{and} \quad \hat{e}_2 = u_2 + (h_1(e_1) + )T.
\end{align*}
\]

Then
\[
\begin{align*}
\| e_1 \| &\leq \| u_1 \| + \| h_2(e_2) + )T \| \\
&\leq \| u_1 \| + \alpha_2 \| e_2 \| + \beta_2 \\
&= \| u_1 \| + \alpha_2 \| u_2 \| + h_1(e_1) + )T \| + \beta_2 \\
&\leq \| u_1 \| + \alpha_2 \| u_2 \| + x_2 \| e_2 \| + x_2 \alpha_1 \| e_1 \| + \beta_2 \\
&\leq \| u_1 \| + \alpha_2 \| u_2 \| + x_2 \| e_2 \| + \beta_2.
\end{align*}
\]

Since \( \alpha_1, \alpha_2 < 1 \) we can write
\[
\| e_1 \| \leq \frac{1}{1 - x_1 \alpha_2} (\| u_1 \| + \alpha_2 \| u_2 \| + x_2 \beta_1 + \beta_2).
\]

Similarly for \( \| e_2 \| \).

**Part 6 is straightforward.**
If \( u_1, \in L^p \) and \( u_2 \in L^q \) then,
\[
\| u_1 + u_1 \|_p \leq \| u_1 \|_p + \| u_2 \|_p \quad \text{and} \quad \| u_2 \|_q \leq \| u_2 \|_q + T > 0.
\]
Hence \( \| e_1 \| \) is bounded uniformly in \( T \Rightarrow e_1 \in L^p \) and \( e_2 \in L^q \).
\[
\| y \|_q \leq \| e_1 \|_p + p, \quad \forall \, T > 0
\]
unifomly in \( T \).
\[
\Rightarrow y_1 \in L^q \quad \text{Similarly} \quad y_2 \in L^p
\]
Remark: We interpret the above result as saying that the feedback system is stable if \( \delta_1, \delta_2 < 1 \).

In the small gain theorem the well-posedness hypothesis H3 appears to be hard to verify. One would like a sufficient condition that would be strong enough to imply this.
The assumption of a stronger hypothesis can ensure that hypothesis H3 on well-posedness holds in fact.

**Definition**
A map $F : L^p \rightarrow L^p$ is said to be incrementally finite gain stable if

(i) $F(0) \in L^p$ where $0$ is the identically zero input.

(ii) For all $T > 0$, $u, v \in L^p$, there exists $k > 0$ such that

$$\frac{\|F(u) - F(v)\|}{\|u - v\|} \leq k \quad \text{for all } T > 0, u, v \in L^p$$

($k$ is independent of $T, u, v$ etc.)

**Lemma:** If $F : L^p \rightarrow L^p$ is causal and incrementally finite gain stable with gain $k < 1$, then there is a unique $u^* \in L^p$ such that $F(u^*) = u^*$.

**Proof:** By hypothesis,

$$\frac{\|F(u) - F(v)\|}{\|u - v\|} \leq k \quad \text{for all } T > 0, u, v \in L^p$$

and $k < 1$. 

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By causality, \( \mathbf{F}_T(u) = F_T(u_T) \).

Here,
\[
\| F_T(u_T) - F_T(v_T) \| \leq k \| u_T - v_T \| + T > 0
\]

But \( \| F(u) - F(v) \| \leq \sup_{T > 0} \| F_T(u) - F_T(v) \| < k \sup_{T > 0} \| u - v \| + u, v \in L_p \)

Then \( F : L_p \to L_p \) the restriction to \( L_p \) is a global contraction. Since \( L_p \) is a Banach space, there is a unique fixed point \( u^* \in L_p \) such that
\[
F(u^*) = u^*
\]

Can there be a \( v^* \in L_p \) but \( v^* \notin L_p \) such that \( F(v^*) = v^* \) (and \( v^* \neq u^* \) necessarily)?

Clean up the argument in 4.17 (Sanity)
Some examples

1. \( H_1 : L_\infty e \rightarrow L_\infty e \)
\[ u \mapsto u^2 \]

is causal but unstable.

2. \( H_1(u)(t) = \int_0^t e^{-a(t-\tau)}u(\tau)\,d\tau \)
\( H_2(u)(t) = k \, u(t) \quad a > 0 \)

\( H_1 : L_\infty e \rightarrow L_\infty e \)
\[ \gamma_1 = \frac{1}{a} \quad \beta_1 = 0 \]

\( H_2 : L_\infty e \rightarrow L_\infty e \)
\[ \gamma_2 = |k| \quad \beta_2 = 0 \]

Small gain theorem says
\[ \frac{1}{a} \, |k| < 1 \quad \Rightarrow \quad \text{stability of closed loop system} \]

This is commensurate with the same that
\[ -a < k < a \]

\( H_2 \) is a necessary and sufficient condition for closed loop stability

\( \text{from the transfer function} \)
\[ G(s) = \frac{1}{s + a} \text{ closed loop} \]