A little detour on measure etc

(see Real Analysis by H. Royden for full scoop)

(a) Any set $\Omega$ has associated to it the set $2^\Omega$ of all subsets of $\Omega$. Consider a subcollection $A \subseteq 2^\Omega$ of subsets of $\Omega$ satisfying

(i) $\Omega \in A$

(ii) $A \in A \Rightarrow A^c$ (the complement of $A$ in $\Omega$) belongs to $A$

(iii) $A_n \in A$, $n=1,2,\ldots$ $\Rightarrow \bigcup_{k=1}^{\infty} A_k \in A$

Such a collection $A$ is called a $\sigma$-algebra of subsets of $\Omega$. Clearly $2^\Omega$ itself is a $\sigma$-algebra.

So is the (very small) collection $\{\emptyset, \Omega\}$ where $\emptyset$ denotes the empty set.

(b) A measureable space is a pair $(\Omega, A)$, where $A$ is a $\sigma$-algebra of subsets of $\Omega$.

(c) A set function $\mu : A \rightarrow [0, \infty] \cup \{+\infty\}$ is a measure if

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{k=1}^{\infty} \mu(A_n)$$

for $A_1, A_2, A_3, \ldots$ a sequence in $A$

such that $A_i \cap A_j = \emptyset$ for $i \neq j$.

If further $\mu(\Omega) < \infty$, then we say that it is a finite measure.

It is clear that $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$
(d) Suppose $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are two measurable spaces. Then \( f : \Omega_1 \to \Omega_2 \) is said to be a measurable mapping, provided \( \mathcal{B} \in \mathcal{A}_2 \Rightarrow f^{-1}(\mathcal{B}) \in \mathcal{A}_1 \).

(e) Given any \( \mathcal{G} \subseteq 2^{\Omega} \), there is a \( \sigma \)-algebra \( \mathcal{A} \) such that \( \mathcal{G} \subseteq \mathcal{A} \subseteq 2^{\Omega} \) and furthermore it is the smallest \( \sigma \)-algebra with this property. We refer to it as the \( \sigma \)-algebra generated by \( \mathcal{G} \), often denoted as \( \mathcal{A} = \sigma(\mathcal{G}) \).

(f) Suppose \( \Omega = \mathbb{R}^1 \) and \( \mathcal{G} = \) collection of all open intervals of \( \mathbb{R}^1 \). Then the \( \sigma \)-algebra generated by \( \mathcal{G} \) is referred to as the Bochner \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}^1) \). It can be shown that every interval, open, closed or semi-open, in \( \mathbb{R}^1 \) is in \( \mathcal{B}(\mathbb{R}^1) \) and is thus a Bochner subset of \( \mathbb{R}^1 \).

(g) We say that \( f : \mathbb{R}^1 \to \mathbb{R}^1 \) is a measurable function if \( f^{-1}(\mathcal{B}) \subseteq \mathcal{B}(\mathbb{R}^1) \) whenever \( \mathcal{B} \subseteq \mathcal{B}(\mathbb{R}^1) \).

(h) Any continuous function is automatically a measurable function.

(i) There is a unique measure, the Lebesgue measure \( \mu : \mathcal{B}(\mathbb{R}^1) \to [0, \infty) \cup \{\infty\} \) such that \( \mu((a,b]) = b-a \).
Lebesgue measure is clearly not a finite measure.

(i) A function \( f : \mathbb{R}^n \to \mathbb{R}^n \) is said to be simple if it is of the form

\[
f(x) = \sum_{i=1}^{N-1} c_i \cdot \mathbb{X}_{\mathcal{E}}(x)
\]

where \(-\infty < x_1 < x_2 < \cdots < x_N < \infty\) \(N\) finite, \(c_i \in \mathbb{R}^n\) and

\[
\mathbb{X}_{\mathcal{E}}(x) = \begin{cases} 1 & x \in [a, b] \\ 0 & \text{otherwise} \end{cases}
\]

(k) The integral \( I(f) \) is defined as

\[
I(f) = \int f(x) \mu(dx)
\]

where \( \mu(dx) \) is the Lebesgue measure can be defined for simple functions as

\[
I(f) = \sum_{i=1}^{N-1} c_i \cdot (x_{i+1} - x_i)
\]

Theorem: If \( \{f_n\} \) is a sequence of simple functions,

\[
f_n(x) = \sum_{k=1}^{N(n) - 1} c_{i_k} \cdot \mathbb{X}_{\mathcal{E}}(x)
\]

and similarly

\[
f_n(x) = \sum_{k=1}^{N(n) - 1} d_{i_k} \cdot \mathbb{X}_{\mathcal{E}}(y)
\]
and \( f_n \leq f_{n+1} \), \( g_n \leq g_{n+1} \) \( n = 1, 2, \ldots \)

and \( f_n \to f \) and \( g_n \to g \), then

\[
\lim_{n \to \infty} I(f_n) = \lim_{n \to \infty} I(g_n)
\]

Using the above result and using the fact that for each measurable function \( f \) there exists \( \{f_n\} \) a sequence of simple functions \( f_n \leq f_{n+1} \) and \( f_n \to f \) as \( n \to \infty \), we can define unambiguously

\[
I(f) = \lim_{n \to \infty} I(f_n)
\]

for any measurable function. This is the Lebesgue integral of \( f \).

(6) Two functions \( f_i : [0, \infty) \to \mathbb{R} \), \( i = 1, 2 \) are said to be equivalent \( f_1 \sim f_2 \) if

\[
\mu(\{x : f_1(x) \neq f_2(x)\}) = 0.
\]

We denote the equivalence class of \( f \) to be \([f]\).

The spaces we define below are all spaces of equivalence classes of measurable
functions. To avoid awkward notation notation, we will continue to use \( f \) when we really mean \([f]\).

(m) The \( L_p \) spaces: \( 1 \leq p < \infty \)

\[ L_p [0, \infty) = \left\{ f : [0, \infty) \to \mathbb{R} \mid f \text{ measurable}, \quad \int_0^\infty |f(x)|^p \mu(dx) < \infty \right\} \]

\[ L_\infty [0, \infty) = \left\{ f : [0, \infty) \to \mathbb{R} \mid f \text{ measurable}, \quad \text{ess sup}_{x \in [0, \infty)} |f(x)| < \infty \right\} \]

(Here, \( \text{ess sup} f = \sup f \) where \( A \) is a suitable set of measure zero.)

**Theorem**

(i) \( L_p \) with \( \|f\|_p = \left( \int |f(x)|^p \mu(dx) \right)^{1/p} \) is complete.

(ii) \( L_\infty \) with \( \|f\|_\infty = \text{ess sup}_{x \in [0, \infty)} |f(x)| \) is complete.

(iii) \( L_2 \) is a Hilbert space with inner product

\[ \langle f, g \rangle = \int_0^\infty f(t) g(t) \mu(dt) \]

and \( \|f\|_2 = \sqrt{\langle f, f \rangle} \).