7 The converse of the stability theorems

The theorems which are discussed in Sections 3 and 4 furnish sufficient conditions. There are, however, no systematic procedures for finding Lyapunov functions. If one does not find a Lyapunov function to prove a particular stability property, one can never conclude that the system under consideration does not have that stability property.

Therefore, in order to have some idea about the usefulness of the technique, it is important to know if the theorems of the preceding sections can be reversed. In other words, if a dynamic system has a particular stability property, can this always be proved by a Lyapunov function? The following theorems are converse theorems.

Theorem 7.1 Assume that \( V(x, t) \) satisfies a Lipschitz condition, and that the origin is an equilibrium state of the dynamic system (4.1). Then:

(i) If the equilibrium state at the origin is uniformly stable, there exist, on some neighborhood of the origin, a positive definite, decreasing function \( W(x, t) \) with a negative semi-definite derivative along the solutions of (4.1);

(ii) If the equilibrium state at the origin is uniformly asymptotically stable, there exist, on some neighborhood of the origin, a positive definite, decreasing function \( W(x, t) \) with a negative definite derivative along the solutions of (4.1);

(iii) If the equilibrium state at the origin is uniformly asymptotically stable in the large, there exists a positive definite, radially unbounded, decreasing function \( W(x, t) \) with a negative definite derivative along the solutions of (4.1).

Theorem 7.1 is mainly of mathematical interest and not very useful from an engineering point of view. Indeed, the theorem proves the existence of a Lyapunov function, but does not yield any procedure to find it. Since the theorem is of limited use, and since the proof is rather lengthy, it is omitted here; interested readers are referred to Hahn (1963).

8 The extent of asymptotic stability

As was pointed out in Chapter 1, asymptotic stability of an equilibrium state of a linear system implies that all motions converge to that equilibrium state. This is not true for non-linear dynamic systems. It is very important to be able to estimate some region about the equilibrium point such that all motions initiating in that region converge to that equilibrium. Only then is it clear whether or not a practical system is stable with respect to the deviations one can expect for the system.
The direct method of Liapunov

The theorem of this section deals with the determination of such a 'domain of attraction' for the autonomous system

\[ \dot{x} = f(x) \]  

(8.1)

with

\[ f(0) = 0. \]

**Theorem 8.1** Let \( V(x) \) be a scalar function. Suppose that the region \( R = \{x : V(x) < d\} \) is bounded. Let \( \dot{V}(x) \) be the derivative of \( V(x) \) along the solutions of (8.1). If \( V(x) \) is positive definite and \( \dot{V}(x) \) negative definite in \( R \), then the origin is an asymptotically stable equilibrium state and all motions starting in \( R \) converge to the origin as \( t \to \infty \).

**Proof** Since \( V(x(t; x_0, t_0)) \) is a non-increasing function of time in \( R \), any solution starting in \( R \) will remain in \( R \) for all time. The argument used in Theorems 4.2 and 4.3 shows the convergence of that solution to the null solution.

As in Section 5, the assumption

\[ \dot{V}(x) < 0 \]

for all \( x \neq 0 \) in \( R \), can be related to the condition: \( V(x) \) is negative semi-definite in \( R \) and does not vanish identically along any solution of (8.1) in \( R \), except the null solution.

From the above theorem the following rule can be derived for the determination of a domain of asymptotic stability. Suppose \( V(x) \) is positive definite for all \( x \), and \( \dot{V}(x) \) is negative definite near the origin. Let \( V_{\text{max}} \) be the lowest value of \( V(x) \) on the surface \( \dot{V}(x) = 0 \). If the region \( R \) about the origin, determined by

\[ V(x) < V_{\text{max}} \]

is bounded, then it is a region of asymptotic stability and belongs to the domain of attraction of the null solution.

The region of asymptotic stability thus obtained clearly depends on the particular Liapunov function. It is not necessarily the largest possible one. It is a subdomain of the total domain of attraction. Two different Liapunov functions may yield two different estimates for the region of asymptotic stability. The union of both regions is an improved estimate for the domain of attraction.

**9 Theorems on instability**

Liapunov's technique also yields theorems on the instability of equilibrium states of dynamic systems; however, these theorems are of limited importance, since one is almost always interested in determining stability properties. However, they are sometimes useful to avoid a waste of effort trying to prove stability.

The following two theorems are examples of instability criteria.
Theorem 9.1 The null solution of the autonomous system

\[ \dot{x} = f(x) \]  

(9.1)
is not asymptotically stable if there exists a scalar function \( V(x) \) with the following properties in some closed neighbourhood \( R \) of the origin:

(i) \( V(x) \) vanishes at the origin, has continuous partial derivatives, and assumes negative values arbitrarily close to the origin; and

(ii) the derivative \( V(x) \) along the solutions of system (9.1) is negative semi-definite.

The null solution is unstable if in addition either \( \dot{V}(x) \) is negative definite or does not vanish identically along any solution of (9.1), except the null solution.

Proof To prove the theorem, it suffices to show that there is at least one solution, starting arbitrarily close to the origin, which is either not asymptotically stable or unstable. For any positive \( \varepsilon \), however small, there is a state \( x_0 \) in \( R \) such that

\[ ||x_0|| < \varepsilon \quad \text{and} \quad V(x_0) < 0. \]

If the motion does not remain in \( R \), then the null solution is not stable. If it remains in \( R \) for all \( t > t_0 \), then

\[ V(t; x_0, t_0) = V(x_0) + \int_{t_0}^{t} \dot{V}(x(t; x_0, t_0)) \, dt < V(x_0) < 0 \]

for all \( t > t_0 \). The motion \( x(t; x_0, t_0) \) does not tend to the origin as \( t \to \infty \).

If the assumptions of the second part of the theorem are satisfied, then the solution \( x(t; x_0, t_0) \) leaves the region \( R \) for some \( t > t_0 \). Indeed, by means of the same argument that was used for the proof of Theorem 5.1 it is possible to show this property. This proves that the null solution is unstable.

Theorem 9.2 Suppose that there exists a scalar function \( V(x) \), with continuous partial derivatives in some neighbourhood \( R \) of the origin, such that:

(i) \( \dot{V}(0) = 0 \), and

(ii) \( \dot{V}(x) \) assumes negative values arbitrarily close to the origin.

If the derivative of \( V(x) \) along the solutions of (9.1) can be expressed as

\[ \dot{V}(x) = a \dot{V}(x) + V'(x) \]

(9.2)

where \( V'(x) \) is negative semi-definite in \( R \), then:

(i) the null solution of (9.1) is not asymptotically stable if \( a \) is a non-negative constant; and

(ii) the null solution of (9.1) is unstable if \( a \) is a positive constant.
Proof Let \( x_0 \) be the initial state that was used in the proof of the preceding theorem. Integrating (9.2) we obtain
\[
P(x(t; x_0, t_0)) < P(x(0)) e^{-\alpha t_0}.
\]
Hence, if \( \alpha \) is a non-negative number, the motion \( x(t; x_0, t_0) \) either leaves \( R \) or does not converge to the origin. If \( \alpha \) is positive, the motion \( x(t; x_0, t_0) \) leaves the region \( R \) for some \( t > t_0 \).
As an example, consider the system
\[
\begin{align*}
x_1 &= -x_1 + x_2 \\
x_2 &= x_1 + x_2 + x_3
\end{align*}
\] (9.3)
The derivative of
\[
P(x) = x_1^2 - x_2^2
\]
along the solutions of (9.3) is
\[
P'(x) = -2x_1 - 2x_2 - x_3.
\]
Theorem 9.1 thus shows that the null solution of (9.3) is unstable.
If in Theorem 5.1, the function \( P(x) \) and \( P'(x) \) are negative definite in \( R \), then the null solution is completely controllable, that is any solution will ultimately leave \( R \) for some \( t > t_0 \).

10 Construction of Liapunov functions

10.1 Survey of available techniques
The main drawback of Liapunov's method for the study of stability of dynamic systems is that there exists no systematic procedure for constructing Liapunov functions. There are, however, some approaches for the construction of Liapunov functions. Some of them are discussed in this section. To investigate the stability of the system
\[
x = f(x, t)
\] (10.1)
one examines the definiteness of the derivative of a positive definite function \( W(x, t) \) given by
\[
W(x, t) = f(x, t) \nabla W(x, t) + \frac{\partial W(x, t)}{\partial t}.
\] (10.2)
There are two basic approaches:
(i) one chooses a positive definite function \( W(x, t) \) and computes the derivative \( W'(x, t) \) by means of (10.2);
(ii) a negative definite function \( W(x, t) \) is selected. The integration of (10.2) yields the Liapunov function \( W(x, t) \).
10.2 Assumption of a Lyapunov function

The simplest method of attacking the problem is to choose a Lyapunov function and to check whether or not the derivative satisfies the conditions of one of the Lyapunov stability or instability theorems. If it does not, no conclusion on the stability behaviour of the dynamic system can be drawn. This merely means that the proposed Lyapunov function is not suited for the problem under consideration. Hence, experience and skill is required to choose suitable functions.

For many problems excellent results can be obtained by using a quadratic form of the state vector \( x \) as part of all the Lyapunov functions. A quadratic form \( V(x) \) of the state vector \( x \) can be expressed as

\[
V(x) = x^T Q x.
\]

The positive definiteness of the function \( V(x) \) is equivalent to the positive definiteness of the matrix \( Q \), for which a necessary and sufficient condition is that the principal minors of the matrix be positive. There exists an interesting method of representing a quadratic form by means of a path integral. It is explained in Appendix B.

A different approach is the "variable gradient" method of Schultz and Gibson (1962) for autonomous dynamic systems. In this approach one assumes the function \( \nabla V(x) \), which contains some adjustable parameters, and computes \( V(x) \) from its gradient and \( V(x) \) from (10.2). The parameters are adjusted, if possible, so that the resulting \( V(x) \) satisfies one of the Lyapunov stability or instability theorems. The following lemma is a useful guide in the selection of the gradient.

**Lemma** A necessary and sufficient condition such that a continuous vector function \( g(x) \) be the gradient of a scalar function is that the matrix

\[
M = \begin{bmatrix}
\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \ldots & \frac{\partial g_1}{\partial x_n} \\
\frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \ldots & \frac{\partial g_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \ldots & \frac{\partial g_n}{\partial x_n}
\end{bmatrix}
\]

(10.4)

be symmetric, where \( g_1, g_2, \ldots, g_n \) denote the components of the vector function \( g(x) \).

**Proof**

(i) The condition is necessary, for if

\[ g(x) = \nabla V(x), \]

then

\[ M = \begin{bmatrix}
\frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \ldots & \frac{\partial V}{\partial x_n} \\
\frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \ldots & \frac{\partial V}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \ldots & \frac{\partial V}{\partial x_n}
\end{bmatrix} \]

is symmetric.
then \( M \) is symmetric, since
\[
\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i}
\]
and
\[
\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}
\]

(i) The same condition is also sufficient. Consider therefore the scalar function
\[
\gamma(x) = \sum_{k=1}^{n} \int_{x_k}^{x_0} g_k(x, 0, \ldots, 0) \, dx_k + \int_{x_k}^{x_0} g_k(x, x_0, 0, \ldots, 0) \, dx_k \\
+ \cdots + \int_{x_k}^{x_0} g_k(x, x_0, \ldots, x_0) \, dx_k
\]
(10.5)

Clearly, since \( M \) is a symmetric matrix,
\[
\frac{\partial^2 V}{\partial x_i \partial x_j} = g_i(x, x_0, \ldots, x_0)
\]
\[
\frac{\partial^2 V}{\partial x_j \partial x_i} = g_j(x, x_0, \ldots, x_0)
\]
\[
\frac{\partial^2 f}{\partial x_i \partial x_j} = g_i(x, x_0, \ldots, x_0)
\]

and hence \( g(x) = \text{grad} \, V(x) \).

Since \( g(x) \cdot dx = \text{grad} \, V(x) \cdot dx = dV(x) \), the integral \( \int_{x_0}^{x} g(x) \cdot dx \) is independent of the particular path joining the points \( A \) and \( B \) of the state space. The function \( V(x) \) can hence be computed from
\[
V(x) = \int_{x_0}^{x} g(x) \cdot dx
\]
(10.6)

along any path joining the origin of the state space to the point \( x \). In (10.5) a particular path was used (Fig. 10.1).

The variable gradient method then consists of selecting a vector function \( g(x) \) such that the matrix \( M \), defined by (10.4), is symmetric. The Liapunov function is computed by means of (10.6), and its derivative from (10.2). The function \( g(x) \) usually contains some adjustable parameters, and one tries to pick them in such a way that the tentative Liapunov function and its derivative satisfy the requirements of a stability or an instability theorem. If this is impossible, the vector \( g(x) \) is altered and new \( V(x) \) and \( \dot{V}(x) \) are evaluated. This approach yields excellent results for many applications.
Consider the second-order system
\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - x_1^2
\end{align*} \]  
(10.7)

Take
g(x) = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} 
(10.8)

where \( a, b, c, \) and \( d \) may be functions of \( x_1 \) and \( x_2 \). The matrix \( M \) is symmetric if for all \( x_1 \) and \( x_2 \),
\[ x_1 \frac{\partial a}{\partial x_1} + b + x_1^2 \frac{\partial b}{\partial x_1} - c + x_1 \frac{\partial c}{\partial x_1} + x_2 \frac{\partial d}{\partial x_1} \]  
(10.9)

The derivative of the tentative Liapunov function is
\[ P(x) = x_1(x_2^2 - c - dx_1) - (d - b)x_2^2 - cx_1^2 \]  
(10.10)

and is clearly negative definite if \( b, c, \) and \( d \) are constant, and if
\[ d > b, \quad c > 0, \quad a = c + dx_1. \]

Then (10.9) yields
\[ b = c, \]
and finally we have
\[ d > b = c > 0. \]  
(10.11)

The remaining step is to compute \( P(x) \) and to check for definiteness. One obtains
\[ P(x) = \frac{dx_1^2}{4} + \frac{cx_1^2}{2} + cx_1x_2 + \frac{dx_2^2}{2} \]
which is positive definite if (10.11) is satisfied. This establishes the global asymptotic stability of the null solution of (10.7).

10.2 Zubov’s method

The second approach is to choose a negative definite $W(x, t)$, and to obtain $W(x, t)$ by integrating the partial differential equation (10.2). If this integration is possible, then the function $W(x, t)$ yields complete information on the stability properties of the equilibrium state at the origin of the state space. The drawback of this method is that the integration of (10.2) is often as difficult as the integration of the dynamic system equation, and sometimes even more difficult. This method is, therefore, of theoretical rather than practical interest.

For autonomous systems this approach is due to the Russian author Zubov (1957). The basic idea of his technique is discussed below, using heuristic arguments.

A negative definite function $Q(x)$ is chosen as the derivative of a Liapunov function along the solutions of the system

$$\dot{x} = f(x).$$

(10.12)

Suppose that it is possible to find the solution $V(x)$ of the partial differential equation

$$\dot{V}(x) = \nabla V(x) f(x)$$

(10.13)

with the boundary condition $V(0) = 0$.

If the equilibrium state $0$ of the system is asymptotically stable, then we expect $V(x)$ to be positive definite in some neighbourhood of the origin. Indeed,

$$V(x(t; x_0, t_0)) - V(x_0) = \int_{t_0}^{t} Q(x(t'; x_0, t_0)) \, dt$$

yields for an initial state $x_0$ within the domain of attraction of the equilibrium state $0$

$$V(x(t)) = \int t_0^{t} Q(x(t'; x_0, t_0)) \, dt$$

which is positive, unless $x_0 = 0$.

If $x_0$ approaches the boundary of the domain of attraction of the null solution, then $V(x_0)$ tends to infinity. This heuristic argument shows that the boundary of the domain of asymptotic stability is found by determining the set of points $x$ in the state space for which the function $V(x)$ becomes infinite.

Generally the procedure is altered to avoid having to find the points where $V(x)$ becomes infinite. A function $Z(x)$, related to $V(x)$, is defined by

$$Z(x) = 1 - \exp[-V(x)].$$

(10.14)
The differential equation (10.13) is thereby transformed into

\[ f(\mathbf{x}^2, \text{grad} Z(\mathbf{x})) = \mathcal{Q}(\mathbf{x})(1 - Z(\mathbf{x})) \]  

(10.15)

with the boundary condition \( Z(0) = 0 \). The boundary of the domain of attraction of the equilibrium state \( \theta \) is found by solving

\[ Z(\mathbf{x}) = 1. \]

**Example**  
Consider the first-order system

\[ x = x^2 - x. \]

Let

\[ \mathcal{Q}(x) = -x, \]

then (10.13) and (10.15) become

\[ \frac{dV}{dx} = -\frac{x}{x^2 - 1} \quad \text{and} \quad \frac{1}{1 - Z} \frac{dZ}{dx} = -\frac{x}{x^2 - 1}. \]

Hence

\[ V(x) = -\ln(1 - x^2), \quad \text{and} \quad Z(x) = x^2. \]

This shows the boundary of the domain of attraction of the equilibrium state to be \( |x| = 1 \).

Rigorous treatment of this technique and methods for numerical solution of the partial differential equation (10.15) can be found in Zubov's book, and in the papers by Margolis and Vogt (1963), and by Szepl (1962). These contributions also contain numerous applications and examples.

**References**


