1.5 Differential Dynamical Systems

In continuous time, we are more used to systems that are described by differential equations. How can we get from $\sum_M$ a differential equation description of the system? To do this we have to make some assumptions about the dependence of $\varphi$ on $t$. The result we are going to describe can be generalized easily to more general input and output spaces.

1.5.1 Theorem: Assume that

a) $U \subseteq \mathbb{R}^m$, $\mathcal{U}$ contains the space of continuous functions from $T$ to $U$ and $X \subseteq \mathbb{R}^n$.

b) for each $t_0, x^0$ and $u(\cdot)$ in $C_m(I)$ (where $I$ is a closed and bounded subinterval of $T$ and $C_m(I)$ is the space of continuous $\mathbb{R}^m$-valued functions on $I$) $\varphi(\cdot; t_0, x^0, u(\cdot))$ is a continuously differentiable function of time from $I$ to $\mathbb{R}^n$ such that the derivative $\frac{d}{dt} \varphi(t; t_0, x^0, u(\cdot))$ is continuous for each $t$, in $t_0, x^0, u(\cdot)$ simultaneously. Then $\varphi(\cdot; t_0, x^0, u(\cdot))$, for $u \in C_m(I)$, is the solution of

$$\begin{align*}
\frac{dx(t)}{dt} &= f(t, x(t), u(t)) \\
x(t_0) &= x^0
\end{align*} \quad \text{for } t_0, t \in I$$

in $C^1_n(I)$ and $f$ is continuous in $t, x, u$.

**Proof:** Left as an exercise.

This gives us the opportunity to discuss some interesting spaces of functions. Denote by $C_m[t_0, t_1]$ the set of $m$-tuples whose elements are continuous functions of time defined on the interval $t_0 \leq t \leq t_1$:

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ \vdots \\ u_m(t) \end{bmatrix}$$

$C_m[t_0, t_1]$ is a vector space over $\mathbb{R}$.

$$\alpha \cdot u(t) = \begin{bmatrix} \alpha u_1(t) \\ \vdots \\ \vdots \\ \alpha u_m(t) \end{bmatrix}; \quad (u + v)(t) = \begin{bmatrix} u_1(t) + v_1(t) \\ \vdots \\ \vdots \end{bmatrix} \quad (1.5.1)$$

How do we measure distance of two functions in $C_m(t_0, t_1)$?
We typically work in function spaces, with distance \( d \) such that if \( d(u(\cdot), u(\cdot)) \to 0 \), as \( n, m \to \infty \), then \( \exists u(\cdot) \) such that \( u^n(\cdot) \to u(\cdot) \).

In \( \mathbb{R}^n \) everybody knows what \( x^n \to x \) means. In \( C[t_0, t_1] \) we have to be careful. So consider a sequence of functions \( u^n(\cdot) \in C[t_0, t_1] \). Then \( u^n(\cdot) \to u(\cdot) \) pointwise if \( \lim_{n \to \infty} u^n(t) = u(t) \) for each \( t \in [t_0, t_1] \). On a set of functions it is important to realize that we can have various modes of convergence (or various topologies). For simplicity consider \( C[0, 1] \). Suppose \( \| x^n \to x^m \|_C \to 0 \) as \( n, m \to \infty \) (i.e. \( \{x^n\} \) is a Cauchy sequence of functions in \( C[0, 1] \)). Fix \( t \in [0, 1] \) then \( \| x^n(t) \to x^m(t) \| \leq \| x^n \to x^m \|_C \) so \( x^n(t) \) converges as a sequence of real numbers, say to \( x(t) \). That is \( x^n(\cdot) \) converge pointwise to \( x(\cdot) \). Now given \( \epsilon > 0 \) choose \( N(\epsilon) \) such that \( \| x^n \to x^m \|_C < \epsilon/2 \) for \( n, m > N(\epsilon) \). Then

\[
| x^n(t) \leftrightarrow x(t) | \leq | x^n(t) \leftrightarrow x^m(t) | + | x^m(t) \leftrightarrow x(t) | \leq \| x^n \leftrightarrow x^m \|_C + | x^m(t) \leftrightarrow x(t) | \tag{1.5.4}
\]

For \( n > N(\epsilon) \) we can choose \( m \) large enough to make \( r.h.s. < \epsilon \). So \( x^n \to x \) uniformly. This is a well known fact about continuous functions, as is the fact that the limiting function \( x \) is also continuous. To see this fix \( \epsilon > 0 \). Then for every \( \delta, t \) and \( n \)

\[
| x(t + \delta) \leftrightarrow x(t) | \leq \| x^n(t + \delta) \to x^n(t) \| + | x^n(t + \delta) \leftrightarrow x^n(t) | + \| x^n(t) \leftrightarrow x(t) |.
\]

The first part of the \( r.h.s. \) is small, we can make the second small by choosing \( \delta \) appropriately and the third by choosing \( n \) large. So \( x \) is continuous. So now we have the following description of a Differential Dynamical System:

\[
\begin{align*}
x(t_0) &= x^0 = \text{given} \\
\frac{dx(t)}{dt} &= f(t, x(t), u(t)) \\
y(t) &= \tau(t, x(t), u(t)) \triangleq h(t, x(t), u(t))
\end{align*}
\tag{1.5.5}
\]

As a result we are interested in looking at differential equations of this type (this is in fact a family of differential equations parametrized by the input). We give now some basic
results about existence, uniqueness of solutions, meaning of solutions in ordinary differential equations of this type. If we substitute \(u(t)\) as a function of time then the above becomes a differential equation of the form

\[
\begin{align*}
  x(t_0) &= x^0 \\
  \frac{dx(t)}{dt} &= F(t, x(t)) 
\end{align*}
\]  

(1.5.6)

For each \(t, F(t, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n\), and assigns a vector to every \(x \in \mathbb{R}^n\) which shows the tangential direction at the corresponding point of the solution of (1.5.6). This is a simple and well known geometric interpretation of (1.5.6). We want to find a curve so that the tangent of the curve differential at \(x(t)\) is \(F(t, x(t))\). \(F(t, \cdot)\) is called a vector field in the language of differential geometry. Here is where control theory and ordinary differential equations theory differ drastically. For the qualitative properties of the dynamical system we need to study the family of vector fields \(f(t, x(t), u(t))\) parametrized by the controls \(u(\cdot)\). When we apply control \(u(\cdot)\) the trajectory follows the tangential direction indicated by \(f(t, x(t), u(t))\). Differential geometric ideas have been applied very successfully to the study of Nonlinear Control Systems in the work of R. Brockett, H. Sussman, A. Krener, R. Hermann and many others. However as far as existence and uniqueness is concerned what needs to be done is to restrict the controls \(u(\cdot)\), so that standard results on existence and uniqueness of differential equations can be applied. One of the best references is J. Hale’s “Ordinary Differential Equations” and in particular Chapter 1. By a local solution of (1.5.6) through \(x^0, t_0\) we mean that we can find an open interval \(I\) containing \(t_0\) and a continuously differentiable function on \(I\) to \(\mathbb{R}^n\) such that (1.5.6) is satisfied.

1.5.2 Theorem: (Peano): If \(F\) is continuous on \(I \times D\), \(I\) an open interval containing \(t_0\), \(D\) an open set containing \(x^0\), then there exists at least one solution of (1.5.6) passing through \((x^0, t_0)\).

Proof: We only give a sketch, the details can be found in Hale’s book. Approximate the differential equation by the difference scheme

\[
x(t + h) = x(t) + F(t, x(t))h
\]

![Figure 1.5.1: Illustrating polygonal approximation used in Peano’s theorem](image)

Copyright ©1980, John S. Baras. All Rights Reserved
Plot the resulting polygonal curves and show that these curves converge as \( h \to 0 \) to a continuous curve (Euler’s method).

Observe that for \( F(t, x) \) continuous (or even piecewise continuous) the differential equation is equivalent to the integral equation \( x(t) = x^0 + \int_0^t F(\tau, x(\tau)) d\tau \) and this is the one that the computer solves! Suppose we have a local solution on \( I \). Can we have a solution on \( I' \supset I \), which agrees with the previous solution on \( I \)? This process is called continuation of solutions. A useful fact to remember here is that if \( F(t, x(t)) \) is bounded on \( I \times D \) then if \( x \) is defined on \((a, b)\), \( x(a+)\), \( x(b-)\) exist.

**1.5.3 Definition:** If we cannot extend the solution further than an interval \( I_{\text{max}} \), we call \( I_{\text{max}} \) the maximal interval of existence.

When \( F \) is continuous, \( \exists \) a maximal interval of existence. If \( I_{\text{max}} \) is \((a, b)\) then \((t, x(t))\) tends to the boundary of \( I_{\text{max}} \times D \) as \( t \to a \) or \( t \to b \).

As an example consider the differential equation

\[
\dot{x}(t) = x^2
\]

and notice that \( x(t) = \frac{1}{t + c} \) for \( c \) real is a solution. Clearly \( I_{\text{max}} = (\frac{-\infty}{c}, c) \) for \( c < 0 \). We say we have a finite escape time.

![Plot of trajectories of quadratic ODE in example](image)

**Figure 1.5.2:** Plots of trajectories of quadratic ODE in example

Whenever we have discontinuous right hand side then we solve piece by piece and patch up the pieces. Piecing together local solutions we may get a global solution.

An example for non-uniqueness is provided by

\[
\dot{x} = \begin{cases} 
\sqrt{x}, & x \geq 0 \\
0, & x < 0 
\end{cases}
\]
Clearly \( x(t) = 0 \) is a solution and
\[
\phi(t) = \begin{cases} 
(t \leftrightarrow c)^2/4, & t \geq c \\
0, & t < c 
\end{cases}
\]
is also a solution. There are two solutions starting from any point \((0, t_0)\), as shown in the figure above.

To obtain uniqueness of solutions we need to make some more assumptions about the right hand side \( F(t, x(t)) \).

1.5.4 Definition: \( F : I \times D \to \mathbb{R}^n \) is locally Lipschitz continuous in \( x \), if for any bounded subset \( U \) of \( I \times D \), \( \exists \) a constant \( k_U \) (which may depend on \( U \)) such that \( \| F(t, x) - F(t, y) \|_U \leq k_U \| x - y \|_U \) for \( (t, x), (t, y) \) in \( U \).

For example if \( F \) has continuous partial derivatives with respect to \( x \) in \( D \), then it is locally Lipschitz continuous. However the function \( \sqrt{x} \) is not. Indeed if \( D \subset \{ x > 0 \} \), i.e. is an open interval around \( x^0 \), then for \( x^1, x^2 \in D \)
\[
| \sqrt{x^1} \leftrightarrow \sqrt{x^2} | = \frac{1}{\sqrt{x^1} + \sqrt{x^2}} | x^1 \leftrightarrow x^2 | \leq \frac{1}{2\sqrt{x^0} \leftrightarrow \delta} | x^1 \leftrightarrow x^2 |
\]
However when \( D \) intersects \( x = 0 \), we cannot achieve such a bound.

1.5.5 Theorem: (Picard-Lindelöf): Assume

a) \( F(t, x) \) is continuous in \( I \times D \)

b) Locally Lipschitz continuous with respect to \( x \) in \( D \).

Then there exists a unique solution \( x(t; t_0, x^0) \) passing through \((t_0, x^0)\). For each \((t_0, x^0) \in I \times D \) we get a local solution, and let \( t_{\max}(t_0, x^0) \) denote the maximum time interval of continuation of each one of these local solutions. Let \( E = \{(t; t_0, x^0) \mid (t_0, x^0) \in I \times D \text{ and } t \in I_{\max}(t_0, x^0)\} \). \( E \) is a subset of \( \mathbb{R}^{n+2} \). Then \( x(t; t_0, x^0) \) is continuous on \( E \).

Notice that the last statement provides continuity of solution with respect to: 1) initial condition and 2) initial time.
Proof: We only give a sketch. The details can be found in Hale’s book. The same arguments work for more general cases, so for illustration purposes we assume $F(t, x(t))$ is scalar valued. Now the differential equation is equivalent to

$$x(t) = x^0 + \int_{t_0}^{t} F(\tau, x(\tau)) d\tau \quad (1.5.7)$$

![Figure 1.5.4: Schematic for Picard-Lindelöf theorem](image)

We can view the r.h.s. as a transformation $T$ which acts on the function $x$ and produces another function. That is suppose $y \in C(I)$ and takes values in $D$, then $T$ is defined via:

$$(Ty)(t) = x^0 + \int_{t_0}^{t} F(\tau, y(\tau)) d\tau \quad (1.5.8)$$

What are we looking for? Find a function $x \in C(I')$, $I'$ a subinterval of $I$, with values in $D$ such that $x = Tx$ (as functions). The important point for the proof is to realize that **WE ARE LOOKING FOR A FIXED POINT** of the transformation $T$. Assumptions made allow us to pick subintervals $I' = [t_0 \leftrightarrow a, t_0 + a]$, $D' = [x^0 \leftrightarrow \beta, x^0 + \beta]$ of $I, D$, containing $(t_0, x^0)$ with the properties:

1) a function $y \in C(I')$ with values in $D'$ has an image under $T$ which is also in $C(I')$ and takes values in $D'$.

2) it is obvious that if $y(t_0) = x^0$ then also $(Ty)(t_0) = x^0$.

Therefore the transformation $T$ sends the subset $B = \{y \in C(I') \mid y(t_0) = x^0 \text{ and } |y(t) - x^0| \leq \beta\}$ into itself. We can show next that:

a) $B$ is a closed subset of $C(I')$, i.e. if $u^n \in B$, $\|u^n - u\|_{C} \to 0 \Rightarrow u \in B$

b) for $y^1, y^2 \in B$, we have

$$\|Ty^1 - Ty^2\|_C \leq \rho \|y^1 - y^2\|_C \quad (1.5.9)$$

using Lipschitz continuity, where $0 < \rho < 1$, by appropriately selecting $\alpha, \beta$. (1.5.9) states that $T$ is a **contraction map**. The rest follows from the contraction mapping principle. We give a simple version of this principle but the theorem holds true for any complete metric space.
1.5.6 Theorem: (Contraction mapping theorem for $C(I)$). Suppose

a) $B$ is a closed subset of $C(I)$

b) $T$ maps $B$ into $B$

c) for $y^1, y^2 \in B, \|Ty^1 \leftrightarrow Ty^2\|_C \leq \rho \|y^1 \leftrightarrow y^2\|_C$

$$0 < \rho < 1$$

Then

1) $T$ has a unique fixed point in $B$, say $y^*$

2) starting from any $y^0$ in $B$ the sequence $y^n = T^ny^0$ will converge to $y^*$, as $n \to \infty$.

Proof: assume $n > m$ without loss of generality. Then

$$\|y^n \leftrightarrow y^m\|_C = \|T(T^{m-1}y^0) \leftrightarrow T(T^{m-1}y^0)\|_C$$

$$\leq \rho \|T^{m-1}y^0 \leftrightarrow T^{m-1}y^0\|_C$$

$$\leq \rho^2 \|T^{m-2}y^0 \leftrightarrow T^{m-2}y^0\|_C$$

$$\leq \rho^m \|T^{m-m}y^0 \leftrightarrow y^0\|_C$$

On the other hand

$$\|y^t \leftrightarrow y^0\|_C = \|y^t \leftrightarrow y^{t-1} + y^{t-1} \leftrightarrow y^{t-2} + \cdots + y^1 \leftrightarrow y^0\|_C$$

$$\leq \|y^t \leftrightarrow y^{t-1}\|_C + \|y^{t-1} \leftrightarrow y^{t-2}\|_C + \cdots + \|y^1 \leftrightarrow y^0\|_C$$

and applying (1.5.10)

$$\leq \rho^{t-1} \|y^1 \leftrightarrow y^0\|_C + \rho^{t-2} \|y^1 \leftrightarrow y^0\|_C + \cdots + \|y^1 \leftrightarrow y^0\|_C$$

$$= (1 + \rho^2 + \cdots + \rho^{t-1}) \|y^1 \leftrightarrow y^0\|_C$$

$$= \frac{1-\rho^t}{1-\rho} \|y^1 \leftrightarrow y^0\|_C$$

(1.5.11)