4. We study properties of the transition matrix associated to the differential equation

\[ \frac{dx(t)}{dt} = A(t) x(t) \]

\[ x(t_0) = x_0. \]

The solution \( x(t) \) is given by

\[ x(t) = \Phi(t, t_0) x_0 \]

where \( \Phi \) is the transition matrix.

(i) \textbf{Abel–Jacobi–Liouville formula:}

\[ \det(\Phi(t, t_0)) = \exp \left[ \int_{t_0}^{t} \text{tr}(A(s)) \, ds \right] \]

\textbf{Proof}: Let \( c_{ij} \) denote the cofactor of the element \( \phi_{ij} \) of the matrix \( \Phi \).

Then \( \det \Phi = \sum_{i=1}^{n} \phi_{ij} c_{ij} \).

Recall, from the definition of a cofactor, that \( \phi_{ij} \) does not appear in \( c_{ij} \). Hence

\[ \frac{\partial}{\partial \phi_{ij}} \det \Phi = c_{ij} \]
By chain rule,

\[
\frac{d}{dt} \det(\Phi(t, t_0)) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \frac{\partial \det(\Phi(t, t_0))}{\partial \Phi_{ij}} \right] \frac{d \Phi_{ij}(t, t_0)}{dt}
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} \frac{d \Phi_{ij}(t, t_0)}{dt}
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}^{T} \frac{d \Phi_{ij}(t, t_0)}{dt}
\]

(where \( C_{ij}^{T} = (j, i)^{th} \) element of the transpose of the matrix \( C \) of cofactors of \( \Phi \).

\[
= \text{tr} \left( C^{T} \frac{d \Phi(t, t_0)}{dt} \right)
\]

\[
= \text{tr} \left( C^{T} A \Phi \right)
\]

\[
= \text{tr} \left( A \Phi C^{T} \right)
\]

\[
= \text{tr} \left( A \det(\Phi) \cdot I \right) \quad \text{(Lecture 13, sec.13)}
\]

\[
= \det(\Phi) \text{ tr} (A)
\]
Integrating the above scalar differential equation, we obtain
\[
\det (\Phi (t, t_0)) = \exp \left( \int_{t_0}^{t} tr(A(s)) \, ds \right) \det (\Phi (t_0, t_0))
\]
But \( \det (\Phi (t_0, t_0)) = 1 \).

Hence
\[
\det (\Phi (t, t_0)) = \exp \left( \int_{t_0}^{t} tr(A(s)) \, ds \right)
\]

(ii) \( \Phi (t, t_0) \) is invertible.

Proof: Under the running hypothesis that elements of \( A(t) \) are continuous functions of \( t \) on the interval \([t_0, t_1]\), it follows that \( \int_{t_0}^{t} tr(A(s)) \, ds \) is a bounded function of \( t \) on the interval \([t_0, t_1]\) and hence its exponential is always positive. From (i) \( \Phi \) is invertible.

(iii) \( \Phi (t, t_0) = \Phi (t, t_1) \Phi (t_1, t_0) \)

Proof: For any \( x_0 \), \( \Phi (t_1, t_0) x_0 \) is the
state $x_1 = x(t_1)$ attained by solving

$$\dot{x} = Ax \quad \text{and} \quad x(t_0) = x_0.$$ 

Treating $x_1$ as initial state at time $t_1$,

$$x(t) = F(t, t_1)x_1 = F(t, t_1)F(t_1, t_0)x_0$$

is the state attained at time $t$ for the same o.d.e.

On the other hand $F(t, t_0)x_0$ is the state at time $t$ for solving the same o.d.e starting at $x_0$ at $t_0$.

By uniqueness of solutions to linear time varying ordinary differential equations with continuous $A(t)$, it follows that

$$x(t) = F(t, t_0)x_0 = F(t, t_1)F(t_1, t_0)x_0.$$ 

Since $x_0$ is arbitrary, the result follows.

Remark. No specific ordering of $t, t_0$ need be specified. The construction of section 3 of Lecture 2(a) (pages 9-16) makes sense whether $t > t_0$ or $t < t_0$.

Remark. Letting $t = t_0$, we see $F(t_1, t_0) = F(t_0, t_1)$. 

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Suppose we change basis in the state space $\mathbb{R}^n$ by a nonsingular transformation $P(t)$ which is time-dependent. In this new basis the state vector at each instant of time is represented by

\[ \tilde{x}(t) = P(t) x(t). \]

Moreover, unless $P$ is differentiable w.r.t. $t$, we cannot write a derivative for $\tilde{x}$. Assuming differentiability of $P$, we obtain

\[
\begin{align*}
\dot{\tilde{x}} &= \dot{P} x + P \dot{x} \\
&= \dot{P} P^{-1} \tilde{x} + P \dot{A} x \\
&= \left( \dot{P} P^{-1} + P \dot{A} P^{-1} \right) \tilde{x}
\end{align*}
\]

Let us denote by $\Phi(t, t_0)$

\[ \dot{P} P^{-1} + P \dot{A} P^{-1} \]

the transition matrix associated to the above differential equation for $\tilde{x}$. Then, for initial condition $\tilde{x}_0 = P(t_0) x_0$ the solution to the equation for $\tilde{x}$ is

\[ \tilde{x}(t) = \Phi(t, t_0) \tilde{x}_0. \]
\[ x(t) = \Phi(t, t_0) P(t_0) X_0 \]

The solution for the differential equation for \( x \) is given by

\[ x(t) = \Phi(t, t_0) X_0 \]

From the formula for \( z(t) \) and from the change of basis we can also write

\[ x(t) = P^{-1}(t) z(t) \]

\[ = P^{-1}(t) \cdot \Phi(t, t_0) \cdot P(t_0) X_0 \]

By uniqueness of solutions (pages 6-9 of lecture 2(a)), we obtain equality of the above two formulas for \( x(t) \), for all \( x_0 \in \mathbb{R}^n \). It follows that the transition matrix for the system determining \( x \) is related to that of \( z \) by

\[ \Phi(t, t_0) = P(t) \Phi(t, t_0) P(t_0) \]

\[ \frac{\partial}{\partial P^{-1} + PA P^{-1}} \]
In the special case where \( A(t) = A \) is a constant, and \( P(t) = P \) a constant, we see that
\[
(t - t_0)PAP^{-1} \quad e^{(t - t_0)A} P^{-1} = P e^{sA} P^{-1},
\]
a result we already know by direct substitution of the similarity transformation in the power series expansion for the matrix exponential
\[
es PAP^{-1} = \sum_{k=0}^{\infty} \frac{(s P AP^{-1})^k}{k!} \quad \text{for } s \in \mathbb{R}
\]

\[
= \sum_{k=0}^{\infty} \frac{P (s A) P^{-1}}{k!}
\]

\[
= P e^{sA} P^{-1}
\]