where \[ G(s) = C (sI - A)^{-1} B, \]

\[ M(s) = K (sI - (A + HC))^{-1} B \]

\[ N(s) = K (sI - (A + HC))^{-1} H \]

In the usual discussion of state-space theory of observer-controller design, we make the following assumptions:

- \([A, B, C]\) is a minimal triple that realizes \(G(s)\);

- \(K\) is such that \((A - BK)\) has spectrum \(\subseteq \mathbb{C}^-\);

- \(H\) is such that \((A + HC)\) has spectrum \(\subseteq \mathbb{C}^-\);

then, the compensator consisting of \(M, N\) and the unity feedback of \(\dot{y}\) is itself stable. Choice of spectrum of \((A + HC)\) governs how fast \(\hat{x}(t)\) converges to \(x(t)\). Choice of spectrum of \((A - BK)\) is a specification of...
the closed loop system behavior.

It is desirable to carry out the whole design process in the frequency domain, skipping the state space realization and finding $K$ and $H$. We do this below, using Figure 3 as a guide.

\[ Y(s) = G_1(s) U(s) \]

\[ U(s) = V(s) - \frac{1}{2} \overline{Y}(s) \]

\[ \overline{W}(s) = M(s) U(s) + N(s) \overline{Y}(s) \]

**SISO assumption**

From here on we consider the single input, single output (SISO) case only, allowing easier algebra.

In that case,

\[ Y(s) = \frac{G_1(s)}{1 + M(s) + N(s) G_1(s)} \overline{W}(s) \]

define the closed loop transfer function

\[ G_c(s) = \frac{G(s)}{1 + M(s) + N(s) G(s)} \]
Let \( G_c(s) = \frac{b(s)}{a(s)} \), as before,

\[ M(s) = \frac{m_u(s)}{m_d(s)} \quad \text{and} \quad N(s) = \frac{n_u(s)}{n_d(s)} \]

where all transfer functions are rational
(strict, proper and reduced (i.e., free of common factors in the numerator-
denominator expressions)).

Also \( a, m_d, n_d \) are monic.

Then, the closed loop transfer function

\[
G_c(s) = \frac{b/a}{1 + \frac{m_u}{m_d} + \frac{n_u}{n_d} \frac{b}{a}}
\]

\[
= \frac{m_d n_d b}{m_d n_d a + m_u n_d a + n_u m_d b}
\]

Suppose there exist \( m_u, m_d, n_u, n_d \)

Such that

\[
m_u n_d a + n_u m_d b = m_d n_d m
\]

for a specified polynomial \( M(s) \)

\[ \deg (M(s)) \leq n-1. \]
Then, clearly, canceling \( m \) and \( n \) from the numerator and denominator, we obtain,

\[
C_c = \frac{b}{a + m}
\]

Specifying \( m \) and solving (*) yields any closed loop poles we might want — i.e., \( m(s) \) in a design specification.

Solving (*) is equivalent to solving

(**) \[
\frac{mu}{m} a + \frac{nu}{n} b = m
\]

\[
\left( \iff Ma + Nb = m \right)
\]

For strictly proper transfer functions \( M, N \).

We solve (***) in three steps.

1. Find \( x(s), y(s) \) \( \deg(x) < \deg(b) \), \( \deg(y) < \deg(a) \) such that \[ x(s)a(s) + y(s)b(s) \equiv 1. \]

This is the BEZOUT step.
(2) Select monic polynomial $S(s)$ with $\deg(S) = n$.

Define $f = x \, m \, s$

$g = y \, m \, s$

Then

$$\frac{f}{S} a + \frac{g}{S} b = (xa + yb)m$$

$$= m$$

Thus $\frac{f}{S}$ and $\frac{g}{S}$ solve $(\star \star)$

but fail to meet the strict properness conditions. In fact, they are polynomials and not rational functions. We remedy this in the next step.

(3) Divide $g$ by $a$ to get

$$g = la + g_o$$

$\deg(g_o) < \deg(a) = n$

Define $f_o = f + lb$
Then \( M = \frac{f_0}{\delta} \), \( N = \frac{g_0}{\delta} \),

\[\text{solve } (\star \star) \text{, and respect the strict proper rationality conditions.}\]

**proof:**

\[M a + N b = \frac{(f_0 a + g_0 b)}{\delta}\]

\[= \frac{(f + lb) a + (g - la) b}{\delta}\]

\[= \frac{f a + gb}{\delta}\]

\[= \frac{(x a + y b) m \delta}{\delta}\]

\[= m\]

\[\deg(f_0 a) = \deg(m \delta - g_0 b)\]

\[\deg(f_0) = \deg(m \delta - g_0 b) - \deg(a) - 16\]
\[ = \deg (ms - g_0 b) - n \]

\[ \deg (g_0) \leq n-1 \quad \text{and} \quad \deg (b) \leq n-1 \]

\[ \Rightarrow \deg (g_0 b) \leq 2n-2 \]

\[ \deg (m) \leq n-1 \quad \text{and} \quad \deg (s) = n \]

\[ \Rightarrow \deg (ms) \leq 2n-1 \]

Thus \( \deg (ms - g_0 b) \leq 2n-1 \)

Hence \( \deg (f_0) \leq n-1 \) \quad \checkmark

\[ < n = \deg (s) \]

Further \( \deg (g_0) < \deg (a) = n = \deg (s) \)

Thus \( f_0 \in M \) and \( g_0 \in N \) are both rational, strict, proper.

We have thus shown that there is an algorithm to produce desired

\[ G_c = \frac{b}{a + m} \quad \text{in three distinct steps} \]
Compensator Design by Bezout

(1) Solve BEZOUT using repeated Euclidean division to obtain
\[ r_{p-1} = \text{g.c.d.} (a, b) = 1 \] and back-substitution.

(2) Select \( S \) monic, \( \text{deg} (S) = n \) and define \( f = x \cdot m \cdot S \);
\[ g = y \cdot m \cdot S. \]

(3) \[ \Rightarrow \text{Divide } g \text{ by } a \]
\[ g = l \cdot a + g_0 \]
\[ \text{deg}(g_0) < \text{deg}(a) = n \]
define
\[ f_0 = f + lb \]
define
\[ M = \frac{f_0}{g} \quad \text{and} \quad N = \frac{g_0}{g} \]

(4) STOP
In practice we would specify
(or get a specification from a customer)
that \( m(s) \) with \( \text{deg} \ (m) \leq n-1 \)
be such that
\[ a(s) + m(s) \]
has all its roots in desired locations in \( \mathbb{C}^- \).

and

we would choose \( s(s) \) monic
of \( \text{deg} = n \) such that all roots
of \( s \) are in desired locations
in \( \mathbb{C}^- \); this determines how
fast the observer estimates the
state.
\[ G_n(s) = \frac{b(s)}{c(s)} = \frac{s+2}{s^3+3} \]

1. (i) \( (s^3+3) = s^2(s+2) + (-2s^2+3) \)

\[ a = s^2(s+2) + (-2s)(s+2) + 4s+3 \]

\[ = s^2(s+2) + (-2s)(s+2) + 4(s+2) - 5 \]

\[ = (s^2 - 2s + 4)(s+2) - 5 \]

Thus \( \gamma_{-1} \gamma_1 = -5 \)

\[ \gamma_1 = a - \gamma_1 b \]

\[ -5 = (\gamma_1) (s^3+3) + (-s^2+2s-4)(s+2) \]

\[ \Rightarrow 1 = (-\frac{1}{5}) (s^3+3) + (\frac{s^2}{5} - \frac{2}{5} s + \frac{4}{5})(s+2) \]

\[ -20 \]
\[
\begin{align*}
    &= x(s) \cdot a(s) + y(s) \cdot b(s) \\
    \text{Thus,} \quad x(s) &= -\frac{1}{5} \\
    y(s) &= \frac{s^2}{5} - \frac{2s}{5} + \frac{4}{5}
\end{align*}
\]

2. Suppose we want (or the customer wants) the closed loop characteristic polynomial to be

\[
p(s) = a(s) + m(s) = (s+1)^3 = s^3 + 3s^2 + 3s + 1
\]

Then \( m(s) = 3s^2 + 3s - 2 \)

Now select \( \delta(s) \) monic deg \( \delta(s) = 3 \)

\[\rightarrow \text{ We choose } \delta(s) = s^3 + 3\]

Then \( f = x \cdot m \cdot \delta = (-\frac{1}{5}) (3s^2 + 3s - 2) (s^3 + 3) \)

\[g = y \cdot m \cdot \delta = (\frac{s^2}{5} - \frac{2s}{5} + \frac{4}{5}) (3s^2 + 3s - 2) (s^3 + 3)\]

3. \[g = k \cdot a + g_0\]

But \( \delta \) divides \( g \) exactly since \( \delta = a = s^3 + 3 \)

So \( g_0 = 0 \) \( \Rightarrow \) \( f_0 = f + k \cdot b \)
\[ \begin{align*}
\frac{f}{l} &= \left( -\frac{1}{5} \right) \left( 3s^2 + 3s - 2 \right) \left( s^3 + 3 \right) \\
&\quad + \left( \frac{s^2}{5} - \frac{2s}{5} + \frac{4}{5} \right) \left( 3s^2 + 3s - 2 \right) \left( s^2 + 2 \right) \\
&= \frac{1}{5} \left( 3s^2 + 3s - 2 \right) \left\{ \frac{(s^2 - 2s + 4)(s^2 + 2)}{s^3 + 3} \right\} \\
&= \frac{1}{5} \left( 3s^2 + 3s - 2 \right) \left( s^3 - 2s^2 + 4s + 2s^2 - 4s + 8 - s^2 - 3 \right) \\
&= \frac{1}{5} \left( 3s^2 + 3s - 2 \right) \left( 3s + s^2 \right) \\
&= 3s^2 + 3s - 2
\end{align*} \]

Thus, \( M = \frac{f_0}{s} = \frac{3s^2 + 3s - 2}{s^3 + 3} \)

\[ N = \frac{g_0}{s} = 0 \]