Problem 1

(a) Consider a linear time-varying system
\[ \dot{x}(t) = A(t) x(t) \]
with \( A(t + T) = A(t) \) for a specific \( T > 0 \) and \( \forall t \in \mathbb{R} \). Under what conditions does this system admit a nontrivial periodic solution of period \( T \)?

(b) Show that the equation
\[ \ddot{x}(t) + \dot{x}(t) + x(t) = \sin(2t) \]
has a unique periodic solution of period \( \pi \). State clearly any results you use.

Problem 2

(a) Consider the linear time-varying system
\[ \begin{align*}
\dot{x}(t) & = A(t) x(t) + B(t) u(t) \\
y(t) & = C(t) x(t)
\end{align*} \]
State clearly what the weighting pattern associated to this system is, and how it determines a relation between input and output.

(b) Under what conditions can a given function \( T(t, \sigma) \) be the weighting pattern of a linear system of the form in (a) above, but with time-invariant coefficients?
(c) Consider the equation
\[ \ddot{y} + 3\dot{y} + 2y = \dot{u} + 2u \]
determining a mapping between scalar signals \( u(t) \) and \( y(t) \). Construct this mapping in terms of a state space realization and associated weighting pattern. Indicate whether your choice above has minimal state space dimension.

**Problem 3**

Consider the system
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
2 & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
1 \\
1
\end{bmatrix} u
\]

(a) When the control is turned off do all solutions of the above system go to 0 as \( t \to \infty \)? Does any solution go to 0 as \( t \to \infty \)?

(b) If your answer to any portion of (a) is NO, how would you proceed to find a state feedback that makes all solutions of the closed loop system go to 0 as \( t \to \infty \)?

(c) If you only have access to an output signal \( y = x_1 \) can you still use feedback to meet the objective of part (b)? Explain steps if your answer is YES.

**Problem 4**

You are given a linear time-invariant system of the form
\[
\begin{align*}
\dot{x} &= Ax + bu \\
y &= cx
\end{align*}
\]
with \( n \) states, 1 input and 1 output. You are also told:

(i) \( \text{Rank} \left( \int_0^1 e^{A\sigma} b \, b^T e^{A^T \sigma} \, d\sigma \right) = n - 1 \)

(ii) \([A, c]\) is observable.

2
Which of the following statements is true?

(a) There is a control of the form \( u(t) = k(t) \ x(t) + v(t) \) such that you can drive the given system from 0 to any final state at time 1.

(b) For any such control as in part (a) you can reconstruct state history over time interval \([0, 1]\) from output history over same time interval.

Your answers must have clearly stated justification.