Problem 1
Consider a linear map $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by the matrix
\[
\begin{bmatrix}
1 & 1 & 2 \\
1 & 0 & 0 \\
2 & 0 & 3
\end{bmatrix}
\]
Compute the matrix representation of this linear map in the basis $\{v_1, v_2, v_3\}$ defined by
\[
v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}.
\]
Does this new matrix have any pure imaginary eigenvalues?

Problem 2
Give a clear and complete definition of the concept of adjoint of a linear map.
Consider
\[
\mathcal{U} = \left\{ u : [0, 2] \rightarrow \mathbb{R}^2 \mid u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \text{ is a continuous function} \right\}
\]
Let $A : \mathcal{U} \rightarrow \mathbb{R}^3$ be the linear map defined as
\[
Au = \begin{bmatrix}
\int_0^2 u_1(r)dr \\
\int_0^2 3u_1(r)dr - \int_0^1 u_2(r)dr \\
-\int_0^{3/2} u_2(r)dr
\end{bmatrix}
\]
Compute the adjoint of $A$. State clearly any hypotheses you need for this purpose.

**Problem 3**

Consider the linear time-varying system

$$\dot{x}(t) = e^{-tA}B e^{tA} x(t)$$

where $A$ and $B$ are constant square matrices. **Show** that a solution to the given system with initial condition $x(t_0) = x_0$ is given by

$$x(t) = e^{-tA} e^{(t-t_0)(A+B)} e^{t_0A} x_0$$

Compute this in the special case,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$B = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}$$

$$t_0 = 1; \quad x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$ 

**Problem 4**

Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

(a) Give a clear justification as to whether or not it is possible to drive the system from $x_1(0) = 1$, $x_2(0) = 2$, to $x_1(1) = x_2(1) = 0$.

(b) If your answer to part (a) is in the affirmative, determine a control that does the prescribed transfer.
Problem 5

Consider a linear time-invariant system of the form

\[
\dot{x} = (A + \epsilon B)x
\]

where \(A\) and \(B\) are known but \(\epsilon\) is a small uncertain parameter. It is of interest to find the sensitivity of the solution at \(t = 1\),

\[
\left. \frac{\partial x(t)}{\partial \epsilon} \right|_{t=1, \epsilon=0},
\]

for a given initial condition \(x(0)\). Solve this problem in the special case,

\[
A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix},
\]

and \(x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\)