Problem 4.1

\[ H_0(z) = 3 + 4z^{-1} - \frac{7}{2}z^{-2} + z^{-3} + z^{-4} \quad \text{with } w = e^{-j\frac{2\pi}{5}} = -1 \]

\[ H_1(z) = H_0(-z) = H_0(Wz) \implies \text{Uniform DFT bank} \]

\[ E_0(z) = 3 - \frac{7}{2}z^{-1} + z^{-2} = (z^{-2}) \left( \frac{3}{\sqrt{5}} - z^{-1} \right) \]

\[ E_1(z) = 4 + z^{-1} \]

Since the causal LTI systems \( w \) transfer functions \( \frac{1}{E_0(z)} \) and \( \frac{1}{E_1(z)} \) are stable (all poles inside \( |z|=1 \))

the following Uniform DFT QMF bank

\[ \begin{array}{c}
\text{Input} \rightarrow E_0(z) \rightarrow W \rightarrow 12 \rightarrow R_0(z^2) \rightarrow z^{-1} \\
\text{Input} \rightarrow E_1(z) \rightarrow W \rightarrow 12 \rightarrow R_1(z^2) \rightarrow z^{-1} \\
\end{array} \]

is PR with \( R_0(z) = \frac{1}{E_0(z)} \) and \( R_1(z) = \frac{1}{E_1(z)} \) (both causal and stable)

In particular \( \hat{x}(m) = 2x(m-1) \) (since \( m=2 \))

Then

\[ F_0(z) = R_0(z^2)z^{-1} + R_1(z^2) = \frac{z^{-1}}{E_0(z^2)} + \frac{1}{E_1(z^2)} \]

\[ = \frac{E_0(z^2) + z^{-1}E_1(z^2)}{E_0(z^2)E_1(z^2)} = \frac{H_0(z)}{E_0(z^2)E_1(z^2)} \quad \text{(Clearly stable)} \]

\[ F_1(z) = -F_0(-z) = \frac{-H_0(-z)}{E_0(z^2)E_1(z^2)} = -\frac{H_1(z)}{E_0(z^2)E_1(z^2)} \]
Problem 4.2

\[ x(z) \xrightarrow{E(z^2)} W \xrightarrow{W^{-1}} M \xrightarrow{W} R_0(z^2) \xrightarrow{z^{-1}} x(z) \]

We clearly have \( H_1(z) = H_0(-z^2) \) and

\[ E_0(z) = R_0(z^2) z^{-1} + R_1(z^2) \quad \text{and} \quad E_1(z) = R_0(z^2) z^{-1} - R_1(z^2) \]

Recall that if we can pick \( R_0(z) \), s.t., \( R_1(z)E_1(z) = S(z) \)

then \( x(z) = z z^{-1} S(z^2) x(z) \) hence almost is eliminated.

(a) We have \( E_0(z) = \frac{A_0(z)}{B_0(z)} \) where \( A_0(z), B_0(z) \) are polynomials in \( z^{-1} \).

Since \( A_0(z) \) has its roots outside the unit circle,

\( \tilde{A}_0(z) = A_0^*(z^{-1}) \) has its roots inside the unit circle.

Furthermore \( |\tilde{A}(e^{j\omega})| = |A^*\left(\frac{1}{e^{j\omega}}\right)| = |A^*(e^{j\omega})| \)

hence \( \frac{A_0(z)}{\tilde{A}_0(z)} \) is all pass.
Here let

\[ R_0(z) = \frac{B_0(z)}{A_0(z)} \quad \frac{A_1(z)}{A_0(z)} \quad R_1(z) = \frac{B_1(z)}{A_0(z)} \quad \frac{A_0(z)}{A_0(z)} \]

Then

\[ R_0(z) E_0(z) = \frac{A_0(z)}{A_0(z)} \quad \frac{A_1(z)}{A_0(z)} = S(z) \]

hence \( S(z) \) is allpass, and since

\[ X(z) = z^{-2} S(z^2) X(z) \]

\[ T(z) \]

\( T(z) \) is also allpass (since \( |S(z^2)| = 1 \)).

The two synthesis filters are given by:

\[ F_0(z) = R_0(z^2) z^{-1} + R_1(z^2) = \]

\[ = \frac{B_0(z^2) A_1(z^2) z^{-1} + B_1(z^2) A_0(z^2)}{A_0(z^2) A_1(z^2)} \]

and \( F_1(z) = \frac{B_0(z^2) A_1(z^2) z^{-1} - B_1(z^2) A_0(z^2)}{\tilde{A}_0(z^2) \tilde{A}_1(z^2)} \)

Right-sided versions of

Clearly, both \( F_0(z) \) & \( F_1(z) \) are stable.
(b) Now write 

\[ E_i(z) = \frac{C_i(z) D_i(z)}{B_i(z)} \]

where \( C_i(z) \) and \( D_i(z) \) are polys in \( z^{-1} \) and 
\( C_i(z) \) has all its roots inside \( |z|=1 \), while \( D_i(z) \) has all its roots outside \( |z|=1 \).

Then, let 

\[ R_0(z) = \frac{B_0(z)}{C_0(z)} \frac{D_i(z)}{D_i(z)} = \frac{C_i(z)}{D_i(z)} B_0(z) \]

\[ R_i(z) = \frac{B_i(z) D_0(z)}{C_i(z) D_i(z) B_0(z)} \]

Clearly, \( R_0(z) \) & \( R_i(z) \) have all their poles inside \( |z|=1 \) (i.e. right-sided and stable), and 

\[ \frac{B_0(z)}{C_0(z)} \frac{D_i(z)}{B_i(z)} = S(z) \]

\[ \text{allpass allpass} \Rightarrow \text{allpass} \]

So 
\[ X(z) = T(z) X(z) \]

where 
\[ T(z) = z^2 S(z^2) \]

and where \( T(z) \) is allpass.

Finally
\[ F_0(z) = \frac{B_0(z) D_i(z) C_i(z) z^{-1} + B_i(z) D_i(z) C_0(z)}{C_0(z) C_i(z) D_i(z) B_i(z)} \]

both stable

& right-sided 

\[ F_i(z) = - F_0(-z) \]
Problem 4.3

(a) \[ X[n] \xrightarrow{Z^{-1}} H_0(z) \xrightarrow{Z} U \xrightarrow{Z} L \xrightarrow{Z} F_0(z) \xrightarrow{Z^{-1}} \hat{X}[n] \]

\[ X_K(z) = z^{-k} H_K(z) X(z) \]

\[ V_K(z) = \frac{1}{2} \left[ X_K(z^{1/2}) + X_K(-z^{1/2}) \right] \]

\[ Y_K(z) = V_K(z^2) = \frac{1}{2} \left[ Y_K(z) + Y_K(-z) \right] = \frac{1}{2} \left[ z^K H_K(z) X(z) + z^{-K} H_K(-z) X(-z) \right] \]

\[ = \left\{ \begin{array}{ll} \frac{1}{2} \left[ H_0(z) X(z) + H_0(-z) X(-z) \right], & k = 0 \\ \frac{z^{1-k}}{2} \left[ H_1(z) X(z) - H_1(-z) X(-z) \right], & k = 1 \end{array} \right. \]

\[ \hat{X}(z) = z^{-1} F_0(z) Y_0(z) + F_1(z) Y_1(z) \]

\[ = \frac{z^{-1}}{2} \left[ F_0(z) H_0(z) + F_1(z) H_1(z) \right] X(z) + \frac{z^{-1}}{2} \left[ F_0(z) H_0(-z) - F_1(z) H_1(-z) \right] X(-z) \]

(b) For \[ H_1(z) = H_0(-z) \quad F_0(z) = H_0(z) \quad F_1(z) = H_1(z) = H_0(-z) \]

we have

\[ A(z) = \frac{z^{-1}}{2} \left[ H_0(z) H_0(-z) - H_0(-z) H_0(z) \right] = 0. \]
(c) \[ T(z) = \frac{z^{-1}}{2} \left[ H_0(z) + H_0(-z) \right] \]

\[ H_0(e^{j\omega}) = e^{-j\omega N/2} A(e^{j\omega}) \text{ real-valued} \]

\[ T(z) = \frac{e^{-j\omega N/2}}{2} \left( e^{-j\omega N} A^2(\omega) + e^{-j(\omega - \pi)N} A^2(\omega - \pi) \right) \]

\[ = \frac{e^{-j(N+1)\omega/2}}{2} \left[ A^2(\omega) + (-1)^N A^2(\omega - \pi) \right] \text{ real.} \]

Hence \( T(z) \) is linear phase.

Since \( |H_0(e^{j\omega})|^2 = A^2(\omega) > 0 \) and \( \theta(z) \) has real coefficients,

\[ |H_0(e^{j\omega})| = |H_0(e^{-j\omega})| \quad \text{implies,} \]

\[ T(e^{j\omega}) = \frac{e^{-j(N+1)\omega/2}}{2} \left[ |H_0(e^{j\omega})|^2 + (-1)^N |H_0(e^{j(\pi - \omega)})|^2 \right] \]

So \( T(e^{j\pi/2}) = \frac{e^{-j(N+1)\pi/4}}{2} |H_0(e^{j\pi/2})|^2 \left[ 1 + (-1)^N \right] \)

\[ = \begin{cases} e^{-j(N+1)\pi/4} |H_0(e^{j\pi/2})|^2, & \text{N even} \\ \phi & \text{N odd} \end{cases} \]

So \( N \) must be even to avoid \( T(e^{j\pi/2}) = 0 \).

(d) Since \( N \) is even, both polyphase components are linear phase.

\[ e_{0}[\omega] = h[N - 2n] = e_0 \left[ \frac{N}{2} - n \right] \]

\[ e_{1}[\omega] = h[N - 2n - 1] = e_1 \left[ \frac{N}{2} + n - 1 \right] \]
Efficient implementation of the analysis base via use of PD and NTs:

\[ H_0(z) = E_0(z^2) + z^{-1} E_1(z^2) \]

\[ H_1(z) = E_0(z^2) - z^{-1} E_1(z^2) \]

Since both \( E_0(z) \) & \( E_1(z) \) are linear phases, computation of each of \( V_0[\text{in}] \) & \( V_1[\text{in}] \) requires \( \frac{N+1}{2} \) multipliers. Hence, for each pair \( E_0(z) \) \& \( V_0[\text{in}] \) we need \( N+2 \) multipliers, or multipliers per unit time in this equal: \( \text{MPU} = \frac{N+1}{2} \).

Alternatively, we may not exploit the NT and get

\[ E_0(z^2) \quad \xrightarrow{z^{-1}} \quad V_0[\text{in}] \]

\[ E_1(z^2) \quad \xrightarrow{-1} \quad V_1[\text{in}] \]

which again require \( \frac{N+1}{2} \) MPU s.
Problem 4.4

Let \( S_i(z) = E_i(z) R_i(z) \)

We have seen that if \( S_i(z) = S(z) \), then

\[
\hat{x}(z) = M z^{-(m-1)} S(z^m) X(z) \quad \text{[no aliasing]}
\]

(a) Let \( R_i(z) = \prod_{j=0}^{m-1} E_j(z) \); clearly causal & FIR.

Then \( S_i(z) = \prod_{j=0}^{m-1} E_j(z) = S(z) \), so

\[
\hat{x}(z) = M z^{-(m-1)} \left[ \prod_{j=0}^{m-1} E_j(z) \right] X(z) \quad \text{(ie alias free)}
\]

(b) The above choice does not lead to a linear phase system function in general (since the \( E_i(z) \) are not assumed to be linear phase).

Note that \( E_c(z) E_k(z) \) is zero phase, but \( E_i(z) \) is non-causal, if \( E_i(z) \) is causal.

However, note that with \( C_i(z) = z^{-N} E_i(z) \),

\[
E_c(z) C_i(z) = z^{-N} E_c(z) E_i(z) \text{ is linear phase,}
\]

and \( C_i(z) \) is causal.

Hence, let \( R_i(z) = \prod_{j=0}^{m-1} E_j(z) \left[ \prod_{j=0}^{m-1} C_j(z) \right] \)}
Then \( S_e(z) = \prod_{j=0}^{M-1} E_j(z) g(z) = z^{-M_N} \prod_{j=0}^{M-1} E_j(z) E_j(z) = S(z) \) for zero phase.

So

\( S(z) \) is linear phase, which also means that \( S(z^m) \) is also linear phase.

And 
\[
X(z) = M - \left[ (M-1) + M_N \right] \left[ \prod_{j=0}^{M-1} E_j(z) E_j(z) \right] X(z)
\]

is linear phase.

**(C)** we have

\( E_t(z) = A_i \prod_{k=0}^{N} b_{t,k}(z) \), where \( b_{t,k}(z) = 1 - a_{t,k} z^{-1} \)

Let

\( C_t(z) = \frac{1}{A_i} \prod_{k=0}^{N} a_{t,k}(z) \)

where

\[
d_{k}(z) = \begin{cases} 
1 - a_{t,k} z^{-1} & \text{if } |a_{t,k}| < 1 \\
1 & \text{if } |a_{t,k}| = 1 \\
a_{t,k}^{-k} z^{-1} & \text{if } |a_{t,k}| > 1
\end{cases}
\]

Then, for \( |a_{t,k}| < 1 \), \( b_{t,k}(z) a_{t,k}(z) = 1 \)

while for \( |a_{t,k}| > 1 \)

\[
|b_{t,k}(z) a_{t,k}(z)| = \frac{1 - a_{t,k} z^{-1}}{a_{t,k}^{-k} z^{-1}} = 1
\]

Hence \(|E_t(z) C_t(z)| = 1\) is all pass.
So, let

\[
R_i(z) = \prod_{j=0}^{m-1} E_j(z) \prod_{j=0}^{m-1} C_j(z)
\]

Then \( S_i(z) = S(z) = \prod_{j=0}^{m-1} [E_i(z) C_i(z)] \) all pass.

Hence \( |S(z^{1/e})| = \prod_{j=0}^{m-1} |E_i(z^{1/e}) C_i(z^{1/e})| = 1 \)

i.e. \( S(z) \) is all pass.

Hence \( \chi(z) = T(z) \chi(z) \)

with \( T(z) = M z^{-(m-1)} S(z^{1/e}) \)

and where

\( |T(z^{1/e})| = M |S(z^{1/e})| = M \) i.e. \( T(z) \) is allpass.
Problem 4.5

We have:

\[ h(z) = E(z^3) e(z), \quad \text{where} \quad e(z) = \begin{bmatrix} 1 \\ z^{-3} - z^{-2} \end{bmatrix}. \]

Similarly:

\[ f(z) = e_p(z) R(z^3), \quad \text{where} \quad e_p(z) = \begin{bmatrix} z^{-2} - z^{-1} \end{bmatrix}. \]

(i) \[ E(z^3) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \quad \det(E(z)) = 1 \] which is of form of \( \z^{-n} \), hence there is an FIR solution for \( R(z) \).

\[ R(z) = E^{-1}(z) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \]

So:

\[ \begin{bmatrix} F_0(z) & F_1(z) & F_2(z) \end{bmatrix} = \begin{bmatrix} z^{1} - z^{-1} \\ -2 + z^{-2} - z^{-1} \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \]

So:

\[ \begin{bmatrix} F_0(z) & F_1(z) & F_2(z) \end{bmatrix} = \begin{bmatrix} 1 - 2z^{-1} + z^{-2} \\ z^{-2} - z^{-1} \\ z^{-1} + z^{-2} \end{bmatrix}. \]

And:

\[ x[l_n] = x_{l_3} - x_{l_1} - x_{l_2}. \]

\[ E(z^3) = \begin{bmatrix} 1 & 0 & z^{-3} \\ 2 & 1 & z^{-3} \\ 3 & 2 & 1 \end{bmatrix}, \quad \det(E(z^3)) = (1 - 2z^{-1}) \] not a scaled delay \( \Rightarrow \) no PR FIR sol for \( R(z) \).
\[ R(z) = E^{-1}(z) = \frac{\text{Adj}(E(z))}{\det(E(z))} = \frac{1}{(1-2z^{-1})} \begin{bmatrix} 1 & 0 & 0 \\ -2+3z^{-1} & 1 & -2z^{-1} \\ -1 & 0 \end{bmatrix} \]


\[
\begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix} = \begin{bmatrix} z^{-2} & z^{-1} & 1 \end{bmatrix} \begin{bmatrix} \frac{1-2z^{-3}}{1-2z^{-3}} & 0 & 0 \\ -2+3z^{-3} & 1 & -2z^{-3} \\ -1 & 0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix} = \begin{bmatrix} 1-2z^{-1}+2z^{-2}+3z^{-4} & 2z^{-5} \end{bmatrix} \begin{bmatrix} 1-2z^{-3} & -2+3z^{-4} & -2z^{-5} \\ -3 & 1 & -2z^{-4} \end{bmatrix}
\]

Since these filters have 3 poles on the unit circle |z| = 2 \Rightarrow 1

To ensure the filters are stable we need to select the filters \( F_2(z) \) that have left-sided impulse responses.

(i.e., these filters cannot be stable & causal and yield PR).

\[
E(z) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & z^{-3} \\ 3 & 1 & 2 \end{bmatrix}
\]

\[ \det(E(z)) = (2 - z^{-1}) \text{ which is not a scaled delay, therefore no IR FIR soln.} \]

\[ R(z) = E^{-1}(z) = \frac{\text{Adj}(E(z))}{\det(E(z))} = \frac{1}{(2-z^{-1})} \begin{bmatrix} 2-z^{-1} & 0 & 0 \\ -4+3z^{-1} & 2 & -2z^{-1} \\ -1 & 0 \end{bmatrix} \]

\[
\begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix} = \begin{bmatrix} z^{-2} & z^{-1} & 1 \end{bmatrix} \begin{bmatrix} \frac{1-2z^{-3}}{2-z^{-3}} & 0 & 0 \\ -1+4z^{-1}+2z^{-2}+3z^{-4} & 2z^{-5} & -4z^{-1} \\ -1 & 0 & 1 \end{bmatrix}
\]

Clearly these filters are FIR, but stable & causal.
Problem 4.6

Clearly the cascade in Figure 3 is equivalent to

\[ X(z) = M \sum_{n=0}^{m-1} (z^{-N})^{(m-1)-n} y_e(z) \]

We have:

\[ \hat{X}(z) = M \sum_{n=0}^{m-1} (z^{-N})^{(m-1)-n} y_e(z) \]

\[ y_e(z) = V_e(z^M) \]

\[ s_e(z) = z^{-NE} X(z) \]

\[ v_e(z) = \frac{1}{M} \sum_{k=0}^{m-1} s_e(z^{-M} w^k) = \frac{1}{M} \sum_{k=0}^{m-1} z^{-NE} w^{-kn} X(z^{-M} w^k) \]

So \[ y_e(z) = \frac{1}{M} \sum_{k=0}^{m-1} z^{-NE} w^{-kn} X(z w^k) \]

Finally,

\[ \hat{X}(z) = M \sum_{n=0}^{m-1} (z^{-N})^{(m-1)-n} \sum_{k=0}^{m-1} z^{-NE} w^{-kn} X(z w^k) \]

\[ = z^{-N(m-1)} \sum_{k=0}^{m-1} X(z w^k) \sum_{l=0}^{m-1} w^{-lNE} = M \]

Clearly, for \[ u = 0 \] \[ \sum_{l=0}^{m-1} w^{-lNE} = M \]
For \( k = 1, \ldots, m-1 \) we have:

\[
\sum_{e=0}^{m-1} w^{-ke} = \sum_{e=0}^{m-1} \left[ w^{ne} \right]^{-k} = 0
\]

But since \( n, m \) are relatively prime,

\[
\{ 1, w^n, w^{2n}, \ldots, w^{n(m-1)} \} \equiv \{ 1, w, w^2, \ldots, w^{m-1} \}
\]

So

\[
\sum_{e=0}^{m-1} w^{-ke} = \sum_{e=0}^{m-1} w^{-ke} = \frac{w^{-km}}{w^{-k}-1} = 0
\]

Hence

\[
\hat{x}(z) = z^{-N(m-1)} M x(z)
\]

i.e.

\[
\hat{x}(z) = M \times [n - N(m-1)] \text{ that is}
\]

the overall system is PR.
Problem 4.7

2) Using the Noble Identities

\[ G_0(z) = H_0(z) H_0(z^3) H_0(z^4) \]

Similarly we can get the rest of the \( G_m(z) \)'s & \( P_m(z) \)'s.

We have

\[ G_0(z) = H_0(z) H_0(z^3) H_0(z^4), \quad P_0(z) = F_0(z^4) F_0(z^3) F_0(z) \]
\[ G_1(z) = H_1(z) H_0(z^3) H_1(z^4), \quad P_1(z) = F_1(z^4) F_0(z^3) F_0(z) \]
\[ G_2(z) = H_0(z) H_1(z^3), \quad P_2(z) = F_1(z^4) F_0(z^3) F_0(z) \]
\[ G_3(z) = H_1(z) H_0(z^2), \quad P_3(z) = F_1(z^4) F_0(z^3) F_0(z) \]
\[ G_4(z) = H_1(z) H_1(z^2) \]

We have:

\[ H_0(z) \quad H_1(z) \quad H_2(z) \]

\[ 0 \quad \frac{\pi}{2} \quad \pi \quad \frac{3\pi}{2} \quad 2\pi \]

For \( G_0(z) \):

\[ 0 \quad \frac{\pi}{2} \quad \pi \quad \frac{3\pi}{2} \quad 2\pi \]

\[ G_0(z) = H_0(z) H_0(z^3) H_0(z^4) \]

For \( G_1(z) \):

\[ 0 \quad \frac{\pi}{2} \quad \pi \quad \frac{3\pi}{2} \quad 2\pi \]

\[ G_1(z) = H_1(z^4) \]

Similarly:

\[ G_2(z), G_3(z), G_4(z) \]

\[ G_0(z), G_1(z), G_2(z), G_3(z), G_4(z) : \text{BWDLTH} \quad \frac{T_0}{\pi} \]

\[ G_0(z), G_1(z), G_2(z), G_3(z), G_4(z) : \text{BWDLTH} \quad \frac{T_0}{2\pi} \]
Figure 7
(b) Since the 2-channel QMF bank is PR, then we can replace it w/ an identity system (i.e., connect its input to the output via "→") (**) 

Clearly: 

1) \( w_1[n] = c_1[n] \)

2) \( b_1[n] = v_1[n] \) (implied by (1) & (**))

3) \( b_2[n] = v_2[n] \) (implied by (**))

4) \( x_1[n] = x_2[n] \) (implied by (2), (3) & (**))

For this of \( w_1[n], c_1[n], b_1[n], v_1[n] \), see attached figure.

(c) The top QMF bank in Fig. 6 can be replaced by a LTI block w/ x̂'s xa function \( T(z) \)

So

(*) \( v_1[n] \) → \[ H_1(z) \rightarrow \{V_2\} \rightarrow T(z) \rightarrow \{V_2\} \rightarrow - \] \( b_1[n] \)

or equivalently

(*) \( v_1[n] \) → \[ H_1(z) \rightarrow \{V_2\} \rightarrow T(z) \rightarrow \{V_2\} \rightarrow - \] \( b_1[n] \)

Alternatively, we can for QMF bank implies

\[ A(z) = \frac{1}{2} [F_0(z)H_0(-z) + F_1(z)H_1(-z)] = 0 \] (***)

&

\[ T(z) = \frac{1}{2} [F_0(z)H_0(z) + F_1(z)H_1(z)] \]
For (*) to be always true we need

\[ A'(z) = \frac{1}{2} \left[ F_0(z) T(z^2) H_0(-z) + F_1(z) H_1(-z) \right] = 0 \quad \text{all } z \]

\[ \iff F_0(z) H_0(-z) \left[ T(z^2) - 1 \right] = 0 \quad \text{all } z \]

\[ \implies T(z^2) - 1 = 0 \iff T(z^2) = 1 \iff T(z) = 1 \]

Hence, (**) will not be always true, and in fact you can easily show that this is the case for the overall system, unless \( T(z) = 1 \).

Observation of (**) shows how we can easily fix the problem: we need a \( T(z^2) \) block at the bottom paddle following \( F_1(z) \), or equivalently a \( T(z) \) block in the lower paddle of (*) following \( \overline{V_2} \). Specifically

\[
\begin{array}{cccccccc}
V_1 & [\overline{V_1}] & \xrightarrow{T_1} & [H_0(z)] & \xrightarrow{V_2} & [F_0(z)] & \xrightarrow{T(z)} & [F_0(z)] \\
& \downarrow \quad [H_1(z)] & \xrightarrow{V_2} & [F_1(z)] & \xrightarrow{T(z)} & [F_1(z)] & \xrightarrow{T(z)} & [F_1(z)] \\
& \downarrow \quad [T(z)] & \xrightarrow{V_2} & [F_1(z)] & \xrightarrow{T(z)} & [F_1(z)] & \xrightarrow{T(z)} & [F_1(z)] \\
& \downarrow \quad \text{equivalent to } \overline{V_1} \iff [T(z^2) T(z)] & \xrightarrow{V_2} & [F_1(z)] & \xrightarrow{T(z)} & [F_1(z)] & \xrightarrow{T(z)} & [F_1(z)] \\
\end{array}
\]

Clearly to make sure that \( \overline{V_1} \iff [T(z^2) T(z)] \rightarrow b_2 \) we simply need to insert \( T(z) \) blocks at both paddles following the \( V_2 \) boxes.
This choice also ensures that

\[ \chi(2) = T(2)T(z^2)T(z^4) \chi(2) \]

i.e. the dual system shown below is alias-free.