Review of Discrete-Time System

Electrical & Computer Engineering
University of Maryland, College Park

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Contact: minwu@umd.edu. Updated: August 28, 2012.
Outline

- Discrete-time signals: $\delta(n)$, $u(n)$, exponentials, sinusoids
- Transforms: ZT, FT
- Discrete-time system: LTI, causality, stability, FIR & IIR system
- Sampling of a continuous-time signal
- Discrete-time filters: magnitude response, linear phase
- Time-frequency relations: FS; FT; DTFT; DFT

**Homework:** Pick up a DSP text and review.
§0.1 Basic Discrete-Time Signals

1. unit pulse (unit sample)

\[ \delta[n] = \begin{cases} 
1 & n = 0 \\
0 & \text{otherwise} 
\end{cases} \]

2. unit step

\[ u[n] = \begin{cases} 
1 & n \geq 0 \\
0 & \text{otherwise} 
\end{cases} \]

Questions:

- What is the relation between \( \delta[n] \) and \( u[n] \)?
- How to express any \( x[n] \) using unit pulses?
  \[ x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k] \]
§0.1 Basic Discrete-Time Signals

3 Sinusoids and complex exponentials

\[ x_1[n] = A \cos(\omega_0 n + \theta) \]
\[ x_2[n] = ae^{j\omega_0 n} \]

- \( x_2[n] \) has real and imaginary parts; known as a single-frequency signal.

4 Exponentials

\[ x[n] = a^n u[n] \text{ (} 0 < a < 1 \text{)} \]
\[ x[n] = a^n u[-n] \]
\[ x[n] = a^{-n} u[-n] \]

Questions:
Is \( x_1[n] \) a single-frequency signal? Are \( x_1[n] \) and \( x_2[n] \) periodic?
The **Z-transform** of a sequence $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.$$ 

In general, the region of convergence (ROC) takes the form of $R_1 < |z| < R_2$.

E.g.: $x[n] = a^n u[n]$: $X(z) = \frac{1}{1-az^{-1}}$, ROC is $|z| > |a|$.

The same $X(z)$ with a different ROC $|z| < |a|$ will be the ZT of a different $x[n] = -a^n u[-n-1]$.
The **Fourier transform** of a discrete-time signal $x[n]$

$$X_{\text{DTFT}}(\omega) = X(z)|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Often known as the Discrete-Time Fourier Transform (DTFT)
- If the ROC of $X(z)$ includes the unit circle, we evaluate $X(z)$ with $z = e^{j\omega}$, we call $X(e^{j\omega})$ the Fourier Transform of $x[n]$
- The unit of frequency variable $\omega$ is radians
- $X(\omega)$ is periodic with period $2\pi$
- The inverse transform is $x[n] = \frac{1}{2\pi} \int_{0}^{2\pi} X(\omega)e^{j\omega n} d\omega$
Question: What is the FT of a single-frequency signal $e^{j\omega_0 n}$?

- Since the ZT of $a^n$ does not converge anywhere except for $a = 0$, the FT for $x[n] = e^{j\omega_0 n}$ does not exist in the usual sense.

- But we can unite its FT as $2\pi \delta_a(\omega - \omega_0)$ for $\omega$ in the range between $0 < \omega < 2\pi$ and periodically repeating, by using a Dirac delta function $\delta_a(\cdot)$. 
Parseval’s Relation

Let $X(\omega)$ and $Y(\omega)$ be the FT of $x[n]$ and $y[n]$, then

$$
\sum_{n=-\infty}^{\infty} x[n] y^*[n] = \frac{1}{2\pi} \int_{0}^{2\pi} X(\omega) Y^*(\omega) d\omega.
$$

i.e., the inner product is preserved (except a multiplicative factor):

$$
<x[n], y[n]> = <X(\omega), Y(\omega)> \cdot \frac{1}{2\pi}
$$

1. If $x[n] = y[n]$, we have $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{0}^{2\pi} |X(\omega)|^2 d\omega$
2. Parseval’s Relation suggests that the energy of $x[n]$ is conserved after FT and provides us two ways to express the energy.

**Question:** Prove the Parseval’s Relation.
(Hint: start with applying the definition of inverse DTFT for $x[n]$ to LHS)
Question 1: How to characterize a general system?
§0.3 (2) Linear Time-Invariant Systems

Suppose

\[
\begin{align*}
\text{(input)} & \quad x_1(n) & \quad \rightarrow & \quad y_1(n) \\
\text{(input)} & \quad x_2(n) & \quad \rightarrow & \quad y_2(n)
\end{align*}
\]

\[
\begin{align*}
\text{suppose} & = & & \text{input} & \rightarrow [T(\cdot)] & \rightarrow \text{output} \\
\frac{\text{input}}{a_1 x_1[n] + a_2 x_2[n]} & \rightarrow \frac{\text{output}}{a_1 y_1[n] + a_2 y_2[n]}
\end{align*}
\]

Linearity

If the output in response to the input \(a_1 x_1[n] + a_2 x_2[n]\) equals to \(a_1 y_1[n] + a_2 y_2[n]\) for every pair of constants \(a_1\) and \(a_2\) and every possible \(x_1[n]\) and \(x_2[n]\), we say the system is linear.

Shift-Invariance (Time-Invariance)

\[
\begin{align*}
\text{(input)} & \quad x_1[n - N] & \rightarrow & \quad y_1[n - N]
\end{align*}
\]

i.e., The output in response to the shifted input \(x_1[n - N]\) equals to \(y_1[n - N]\) for all integers \(N\) and all possible \(x_1[n]\).
An LTI system is both linear and shift-invariant. Such a system can be completely characterized by its impulse response $h[n]$: 

$\text{(input) } \delta[n] \rightarrow \text{(output) } h[n]$ 

Recall all $x[n]$ can be represented as $x[n] = \sum_{m=-\infty}^{\infty} x[m] \delta[n - m]$ 

$\Rightarrow$ By LTI property: 

$y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n - m]$
The input-output relation of an LTI system is given by a convolution summation:

\[ y[n] = h[n] * x[n] = \sum_{m=-\infty}^{\infty} x[m] h[n - m] = \sum_{m=-\infty}^{\infty} h[m] x[n - m] \]

- The transfer-domain representation is \( Y(z) = H(z)X(z) \), where

\[ H(z) = \frac{Y(z)}{X(z)} = \sum_{n=-\infty}^{\infty} h[n] z^{-n} \]

is called the **transfer function** of the LTI system.
A major class of transfer functions we are interested in is the rational transfer function:

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{N} b_k z^{-k}}{\sum_{m=0}^{N} a_m z^{-m}}$$

- \{a_n\} and \{b_n\} are finite and possibly complex.
- \(N\) is the order of the system if \(B(z)/A(z)\) is irreducible.
0.3 (6) Causality

The output doesn’t depend on future values of the input sequence. (important for processing a data stream in real-time with low delay)

An LTI system is causal iff $h[n] = 0 \ \forall \ n < 0$.

**Question:** What property does $H(z)$ have for a causal system?

**Pitfalls:** note the spelling of words “casual” vs. “causal”.
A causal $N$-th order finite impulse response (FIR) system can have its transfer function written as $H(z) = \sum_{n=0}^{N} h[n]z^{-n}$.

A causal LTI system that is not FIR is said to be IIR (infinite impulse response).

e.g. exponential signal $h[n] = a^n u[n]$: its corresponding $H(z) = \frac{1}{1-az^{-1}}$. 
BIBO: bounded-input bounded-output

An LTI system is BIBO stable iff $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

i.e. its impulse response is absolutely summable.

This sufficient and necessary condition means that ROC of $H(z)$ includes unit circle: $|H(z)|_{z=e^{j\omega}} \leq \sum_n |h[n]| \times 1 < \infty$

If $H(z)$ is rational and $h[n]$ is causal (s.t. ROC takes the form $|z| > r$), the system is stable iff all poles are inside the unit circle (such that the ROC includes the unit circle).
We use the subscript “a” to denote continuous-time (analog) signal and drop the subscript if the context is clear.

The **Fourier Transform** of a continuous-time signal $x_a(t)$

\[
X_a(\Omega) \triangleq \int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t} \, dt \quad \text{“projection”}
\]

\[
x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(\Omega)e^{j\Omega t} \, d\Omega \quad \text{“reconstruction”}
\]

- $\Omega = 2\pi f$ and is in radian per second
- $f$ is in Hz (i.e., cycles per second)
§0.4 (2) Sampling

Consider a sampled signal \( x[n] \triangleq x_a(nT) \).

- \( T > 0 \): sampling period; \( 2\pi / T \): sampling (radian) frequency

The Discrete Time Fourier Transform of \( x[n] \) and the Fourier Transform of \( x_a(t) \) have the following relation:

\[
X(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(\Omega - \frac{2\pi k}{T}) |_{\Omega = \omega / T}
\]
If $X_a(\Omega) = 0$ for $|\Omega| \geq \frac{\pi}{T}$ (i.e., band limited), there is no overlap between $X_a(\Omega)$ and its shifted replicas.

Can recover $x_a(t)$ from the sampled version $x[n]$ by retaining only one copy of $X_a(\Omega)$. This can be accomplished by interpolation/filtering.

Otherwise, overlap occurs. This is called **aliasing**.

Reference: Chapter 7 “Sampling” in Oppenheim et al. Signals and Systems Book
Let $x_a(t)$ be a band-limited signal with $X_a(\Omega) = 0$ for $|\Omega| \geq \sigma$, then $x_a(t)$ is uniquely determined by its samples $x_a(nT)$, $n \in \mathbb{Z}$, if the sampling frequency $\Omega_s \triangleq \frac{2\pi}{T}$ satisfies $\Omega_s \geq 2\sigma$.

In the $\omega$ domain, $2\pi$ is the (normalized) sampling rate for any sampling period $T$. Thus the signal bandwidth can at most be $\pi$ to avoid aliasing.
**0.5 Discrete-Time Filters**

1. A Digital Filter is an LTI system with rational transfer function. The frequency response $H(e^{j\omega})$ specifies the properties of a filter:

   $$H(\omega) = |H(\omega)|e^{j\phi(\omega)}$$

   - $|H(\omega)|$: magnitude response
   - $\phi(\omega)$: phase response

2. **Magnitude response** determines the type of filters:

   ![Magnitude response types](image)

3. **Linear-phase filter**: phase response $\phi(\omega)$ is linear in $\omega$.
   - Linear phase is usually the minimal phase distortion we can expect.
   - A real-valued linear-phase FIR filter of length $N$ normally is either symmetric $h[n] = h[N - n]$ or anti-symmetric $h[n] = -h[N - n]$. 

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### 0.6 Relations of Several Transforms

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<th>FREQUENCY-DOMAIN (Synthesis)</th>
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<td>Fourier Series (FS)</td>
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<td>Discrete Fourier Transform (DFT)</td>
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**Fourier Series (FS)**

A continuous periodic function $x(t)$ can be expressed as a Fourier series:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi nt/T}$$

where $T$ is the period of $x(t)$, and $X_n$ are the Fourier coefficients.

**Fourier Transform (FT)**

A continuous aperiodic function $x(t)$ can be transformed to the frequency domain using the Fourier Transform:

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

**Discrete-Time Fourier Transform (DTFT)**

A discrete aperiodic sequence $x[n]$ can be transformed to the discrete frequency domain using the Discrete-Time Fourier Transform (DTFT):

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \text{ for } -\pi < \omega < \pi$$

**Discrete Fourier Transform (DFT)**

A discrete periodic sequence $x[n]$ can be transformed to the discrete frequency domain using the Discrete Fourier Transform (DFT):

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

where $W_N = e^{-j2\pi nk/N}$ is the twiddle factor.
### §0.6 Relations of Several Transforms

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<td>discrete aperiodic</td>
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Discrete-Time Systems

**Question 1:** How to characterize a general system?

*Ans:* by its input-output response (which may require us to enumerate all possible inputs, and observe and record the corresponding outputs)

**Question 2:** Why are we interested in LTI systems?

*Ans:* They can be completely characterized by just one response - the response to impulse input