Multi-rate Signal Processing

3. The Polyphase Representation

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Polyphase Representation: Basic Idea

Example: FIR filter $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$

Group even and odd indexed coefficients, respectively:
$\Rightarrow H(z) = (1 + 3z^{-2}) + z^{-1}(2 + 4z^{-2})$,

More generally: Given a filter $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$, by grouping the odd and even numbered coefficients, we can write

$$H(z) = \sum_{n=-\infty}^{\infty} h[2n]z^{-2n} + z^{-1} \sum_{n=-\infty}^{\infty} h[2n + 1]z^{-2n}$$
Polyphase Representation: Definition

\[ H(z) = \sum_{n=-\infty}^{\infty} h[2n]z^{-2n} + z^{-1}\sum_{n=-\infty}^{\infty} h[2n + 1]z^{-2n} \]

Define \( E_0(z) \) and \( E_1(z) \) as two polyphase components of \( H(z) \):

\[ E_0(z) = \sum_{n=-\infty}^{\infty} h[2n]z^{-n}, \]
\[ E_1(z) = \sum_{n=-\infty}^{\infty} h[2n + 1]z^{-n}, \]

We have

\[ H(z) = E_0(z^2) + z^{-1}E_1(z^2) \]

- These representations hold whether \( H(z) \) is FIR or IIR, causal or non-causal.
- The polyphase decomposition can be applied to any sequence, not just impulse response.
FIR and IIR Example

1. **FIR filter**: \( H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} \)

   \[ \therefore H(z) = (1 + 3z^{-2}) + z^{-1}(2 + 4z^{-2}), \]

   \[ \therefore E_0(z) = 1 + 3z^{-1}; \quad E_1(z) = 2 + 4z^{-1} \]

2. **IIR filter**: \( H(z) = \frac{1}{1-\alpha z^{-1}}. \)

   Write into the form of \( H(z) = E_0(z^2) + z^{-1}E_1(z^2): \)

   \[ \therefore H(z) = \frac{1}{1-\alpha z^{-1}} \times \frac{1+\alpha z^{-1}}{1+\alpha z^{-1}} = \frac{1+\alpha z^{-1}}{1-\alpha^2 z^{-2}} \]

   \[ = \frac{1}{1-\alpha^2 z^{-2}} + z^{-1} \frac{\alpha}{1-\alpha^{-2} z^{-2}} \]

   \[ \therefore E_0(z) = \frac{1}{1-\alpha^2 z^{-1}}; \quad E_1(z) = \frac{\alpha}{1-\alpha^{-2} z^{-1}} \]

   (For higher order filters: first write in the sum of 1st order terms)
Extension to $M$ Polyphase Components

For a given integer $M$ and $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$, we have:

$$H(z) = \sum_{n=-\infty}^{\infty} h[nM]z^{-nM} + z^{-1}\sum_{n=-\infty}^{\infty} h[nM + 1]z^{-nM} + \ldots + z^{-(M-1)}\sum_{n=-\infty}^{\infty} h[nM + M - 1]z^{-nM}$$

Type-1 Polyphase Representation

$$H(z) = \sum_{\ell=0}^{M-1} z^{-\ell} E_\ell(z^M)$$

where the $\ell$-th polyphase components of $H(z)$ given $M$ is

$$E_\ell(z) \triangleq \sum_{n=-\infty}^{\infty} e_\ell[n]z^{-n} = \sum_{n=-\infty}^{\infty} h[nM + \ell]z^{-n}$$

Note: $0 \leq \ell \leq (M - 1)$; strictly we may denote as $E^{(M)}_\ell(z)$. 
Example: $M = 3$

$e_q[n]$

$e_1[n]$

$e_2[n]$

$e_3[n]$

$h[n]$

$z^\ell$: time advance

(there is a delay term when putting together the polyphase components)
Alternative Polyphase Representation

If we define \( R_\ell(z) = E_{M-1-\ell}(z), \quad 0 \leq \ell \leq M - 1 \), we arrive at the **Type-2 polyphase representation**

\[
H(z) = \sum_{\ell=0}^{M-1} z^{-(M-1-\ell)} R_\ell(z^M)
\]

**Type-1**: \( E_k(z) \) is ordered consistently with the number of delays in the input

**Type-2**: reversely order the filter \( R_k(z) \) with respect to the delays
Issues with Direct Implementation of Decimation Filters

Decimation Filters: \[ H(z) \rightarrow \downarrow M \]

**Question:** Any wasteful effort in the direct implementation?

- The filtering is applied to all original signal samples, even though *only every* $M$ filtering output is retained finally.
- Even if we let $H(z)$ operates only for time instants multiple of $M$ and idle otherwise, all multipliers/adders have to produce results *within one step of time.*
- Can $\downarrow M$ be moved before $H(z)$?
  Only when $H(z)$ is a function of $z^M$, we can apply the noble identities to switch the order.
Efficient Structure for Decimation Filter

Apply Type-1 polyphase representation:

\[
\begin{array}{c}
\begin{align*}
\mathcal{M} = &\; E_0(\mathcal{M}) \quad \vdots \\
\mathcal{M} = &\; E_{m-1}(\mathcal{M}) \\
\mathcal{M} = &\; E_{m}(\mathcal{M}) \\
\end{align*}
\end{array}
\]

\[
\begin{array}{c}
\begin{align*}
\text{M-fold decimation} &\; \quad \text{noble identity} \\
\text{MPU} = &\; \frac{N}{M} , \; \text{APU} = \frac{N-1}{M} \\
\end{align*}
\end{array}
\]

\[
\begin{array}{c}
\begin{align*}
\text{filtering are performed at a lower data rate}
\end{align*}
\end{array}
\]
Computational Cost

For FIR filter $H(z)$ of length $N$:

- Total cost of $N$ multipliers and $(N - 1)$ adders is unchanged.
- Considering multiplications per input unit time (MPU) and additions per input unit time (APU), $E_k(z)$ now operates at a lower rate: only $N/M$ MPU and $(N - 1)/M$ APU are required.
- This is as opposed to $N$ MPU and $(N - 1)$ APU at every $M$ instant of time and system idling at other instants, which leads to inefficient resource utilization.
  (i.e., requires use fast additions and multiplications but use them only $1/M$ of time)
Polyphase for Interpolation Filters

Observe: the filter is applied to a signal at a high rate, even though many samples are zero when coming out of the expander.

Using the Type-2 polyphase decomposition:

\[ H(z) = z^{-1}R_0(z^2) + R_1(z^2): \]

- 2 polyphase components
- \( R_k(z) \) is half length of \( H(z) \)

The complexity of the system is \( N \) MPU and \((N - 2)\) APU.
General Cases

In general, for FIR filters with length $N$:

**$M$-fold decimation:**

$$\text{MPU} = \frac{N}{M}, \quad \text{APU} = \frac{N-1}{M}$$

Filtering is performed at a lower data rate

**$L$-fold interpolation:**

$$\text{MPU} = N, \quad \text{APU} = N - L$$

$$\text{APU} = \left( \frac{N}{L} - 1 \right) \times L$$
Fractional Rate Conversion

- Typically $L$ and $M$ should be chosen to have no common factors greater than 1 (o.w. it is wasteful as we make the rate higher than necessary only to reduce it down later).

- $H(z)$ filter needs to be fast as it operates in high data rate.

- The direct implementation of $H(z)$ is inefficient:
  - there are $L - 1$ zeros in between its input samples
  - only one out of $M$ samples is retained
Example: $L = 2$ and $M = 3$

1. Use Type-1 polyphase decomposition (PD) for decimator:

2. Use Type-2 PD for interpolator:
Example: $L = 2$ and $M = 3$

- Try to take advantage of both:

  **Question:** What’s the lowest possible data rate to process? $f/M$

  **Challenge:** Can’t move $\uparrow 2$ further to the right and $\downarrow 3$ to the left across the delay terms.
Trick to enable interchange of $\uparrow L$ and $\downarrow M$

\[ z^{-1} = z^{-3} \cdot z^2 \]

- $z^{-3}$ and $z^2$ can be considered as filters in $z^{-M}$ and $z^{+L}$
- Noble identities can be applied:

\[ \begin{align*}
    z^{-1} &= z^{-3} \cdot z^2 \\
    z^2 &\quad \text{(up)} \\
    z^{-3} &\quad \text{(down)} \\
    z^{-1} &\quad \text{(up)}
\end{align*} \]

\[ \begin{align*}
    z &\quad \text{(down)} \\
    z^{-1} &\quad \text{(up)}
\end{align*} \]

can be interchanged as they are relatively prime
Overall Efficient Structure

Now it becomes

Finally,

\[ R_0(z) = R_{00}(z^3) + z^{-1}R_{01}(z^3) + z^{-2}R_{02}(z^3) \]

\[ R_1(z) = R_{10}(z^3) + z^{-1}R_{11}(z^3) + z^{-2}R_{12}(z^3) \]
Observations

- For $N$-th order $H(z)$: $\text{MPU} = (N + 1)/M \Rightarrow$ independent of $L$

- The final structure is the most efficient:
  - Decimators are moved to the left of all computational units
  - Expanders are moved to the right of all computational units
  Thus the computation is operated at the lowest possible rate.

- The above scheme works for arbitrary integers $L$ and $M$ as long as they are relatively prime.

Under this condition, we have:

1. $\exists n_0, n_1 \in \mathbb{Z}$ s.t. $n_1 M - n_0 L = 1$ (Euclid’s theorem)
   We can then decompose $z^{-1} = z^{n_0 L} z^{-n_1 M}$

2. $\uparrow L$ and $\downarrow M$ are interchangeable
Commutator Model: A Delay Chain followed by Decimators

Polyphase implementation is often characterized by

1. A delay chain followed by a set of decimators,
Commutator Model: Expanders followed by A Delay Chain

A set of expanders followed by a delay chain

Commutator model is an appealing conceptual tool to visualize these operations
Discussions: Linear Periodically Time Varying Systems

Some multirate systems that we have seen are linear periodically time varying (LPTV) systems.

\[ y[n] = \begin{cases} 
  x[n] & \text{if } n \text{ is multiple of } M \\
  0 & \text{otherwise} 
\end{cases} 
= x[n] \cdot c[n] \]

\( c[n] \) is a comb function: takes 1 for \( n \) is multiple of \( M \) and 0 o.w.

\( \Rightarrow \) This is a linear system with periodically time varying response coefficients, and the period is \( M \).
Even though $\uparrow L$ and $\downarrow M$ are time-varying, a cascaded system having them as building blocks may become time-invariant.

This structure is the same as a fractional decimation system with $L = M$. 
Time-invariant System with $\uparrow M$ & $\downarrow M$

Recall: $[X(z)]_{\downarrow M} = \frac{1}{M} \sum_{k=0}^{M-1} X(W_M^k z^{1/M})$
Perfect Reconstruction (PR) Systems

- The above system is said to be a **perfect reconstruction** system if \( \hat{x}[n] = cx[n - n_0] \) for some \( c \neq 0 \) and integer \( n_0 \), i.e., the output is identical to the input, except a constant multiplicative factor and some fixed delay.

- Look ahead: we’ll see the quadrature mirror filter bank (QMF) is generally a LPTV system, reduces to an LTI system when aliasing is completely cancelled, and achieves PR for certain analysis/synthesis filters.
Special Time-invariant System with $\uparrow M$ & $\downarrow M$

Recall: $[X(z)]_{\downarrow M} = \frac{1}{M} \sum_{k=0}^{M-1} X(W_M^k z^{1/M})$

$$\mathbb{Y}(z) = [X(z^M)H(z)]_{\downarrow M} = \frac{1}{M} \sum_{k=0}^{M-1} X(W_M^k z^{1/M}) H(W_M^k z^{1/M}) = X(z)[H(z)]_{\downarrow M}$$

$[H(z)]_{\downarrow M}$ implies decimating the impulse response $h[n]$ by $M$-fold, corresponding to the 0-th polyphase component of $H(z)$.

$\Rightarrow \mathbb{Y}(z) = X(z)E_0(z)$, i.e., an LTI system.